The Orientable Genus of Bipartite Graphs and their Products

Marietta Galea

Supervised by Prof. John Baptist Gauci

Department of Mathematics
Faculty of Science
University of Malta

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To My Family and Philip,

for your constant encouragement and support.

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Abstract

Marietta Galea, BSc. (Hons.)
Department of Mathematics, June 2024
University of Malta

The drawing of graphs on surfaces knows its birth in the 19th century, mainly with the four colour problem and the Heawood map colouring problem. The former problem claims that four colours are enough to properly colour the vertices of a planar graph (that is, in such a way that no two adjacent vertices receive the same colour), whereas the latter one asks for the minimum number of colours required to properly colour the vertices of a graph which is embeddable on a higher-order surface.

A graph is embeddable on a surface if it can be drawn on that surface without edge crossings, and we say that the graph is planar if the surface used for the embedding is the sphere. There are two complementary perspectives that can be considered when embedding graphs on surfaces, namely fixing the graph and determining which surface allows an embedding of that graph, or fixing the surface and determining whether a graph is embeddable on that surface. In topological graph theory, we only consider a surface which is compact and without a boundary. A surface is characterised by whether it is orientable or nonorientable and by its genus. The sphere is the orientable surface with genus 0 and is denoted by S_0. The genus g of an orientable surface S_g is the number of handles that are added to S_0, whereas the genus $\tilde{g}$ of a nonorientable surface $N_{\tilde{g}}$ is the number of Möbius bands (or crosscaps) that are added to S_0. For instance, the torus is the surface S_1 whereas the projective plane is the surface N_1.

For a graph G, the genus of G is the smallest genus of a surface which allows an embedding of G. More formally, a graph G has orientable genus g if it can be embedded on an orientable surface of genus g but not on an orientable surface of genus $g - 1$ (and similarly for the nonorientable genus of a graph). Determining the (orientable and nonorientable) genus of a graph is known to be NP-complete. A lot of work has been done to determine the genus of various families of graphs. Two of the initial results on these lines are given by Ringel and Youngs (1968) and by Ringel (1965), respectively determining the genus of the complete graphs and of the complete

bipartite graphs. Since these initial results, other families of graphs were investigated for their genus, including the products of graphs belonging to the two aforementioned families.

In this dissertation we find, collate, review and analyse the main results dispersed in literature on the orientable genus of bipartite graphs, the Cartesian product and the direct product of two or more bipartite graphs.

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Introduction

1.1 | Preliminaries

A graph G is defined as a pair (V, E), where $V = V(G)$ is a finite non-empty set of *vertices* and $E = E(G)$ is a finite set of unordered pairs of distinct vertices called *edges*. In this work we shall only consider simple graphs, which are graphs without loops or multiple edges. The number of vertices in a graph is denoted by n and the number of edges is denoted by m.

Vertices u and v are said to be *adjacent* if they are connected by an edge $e = \{u, v\}$, sometimes denoted by $e = uv$ for ease of notation. A vertex and an edge are said to be *incident* to each other if the vertex is an endpoint of the edge. The *degree of a vertex v* in a graph G, denoted by $deg_G(v)$, is the number of edges incident to it. A graph is called *regular* if all of its vertices have the same degree. The number $\delta(G) := \min\{deg_G(v) : v \in V(G)\}$ is the *minimum degree* of a graph G, whilst the number $\Delta(G) := \max\{deg_G(v) : v \in V(G)\}$ is the *maximum degree* of G. Unless otherwise stated, we adopt definitions found in [4, 11, 12].

The following result describes the relationship between the sum of degrees of vertices of a graph and the number of edges.

Lemma 1.1.1. *(The Handshaking Lemma) For a graph G, with n vertices labelled $v_1, v_2, \ldots, v_n$ and m edges, the following holds,*

$$\sum_{i=1}^{n} deg_G(v_i) = 2m.$$

A *walk* in G is a sequence of vertices $v_1, v_2, \ldots, v_n$, not necessarily distinct, such that $v_i v_{i+1} \in E(G)$ for $i \in \{1, \ldots, n-1\}$. If $v_1 = v_n$, then it is referred to as a *closed walk*. A *path* in a graph G is a walk whose vertices are all distinct. Then, the *path graph*, denoted by P_n, is the graph whose vertex-set and edge-set are the same as those of a path on n vertices. That is, $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : i \in \{1, \ldots, n-1\}\}$. The *endpoints* of a path of length n are the vertices v_1 and v_n, for $n \geq 2$. A *cycle* in a graph is a closed walk with no repeated edges and no repeated vertices, except for the first and last vertex. Then, a *cycle graph*, denoted by C_n, is the graph whose vertex-set and edge-set are the same as those of a cycle on n vertices. Therefore, $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : i \in \{1, \ldots n\}$ under addition modulo $n\}$. A graph is said to be *connected* if there is a path between every pair of vertices.

If G and H are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G. Moreover, if $V(H) = V(G)$, then H is called a *spanning subgraph* of G. The subgraph $\langle S \rangle$ *induced* by a non-empty set S of vertices of G is the subgraph whose vertex-set is S and whose edge-set consists of all those edges of G that join two vertices in S. The *deletion of a vertex v* in a graph G, denoted by $G - v$, is the subgraph obtained by removing v and all of its incident edges. In general, if S is some set of vertices in G, then $G - S$ is the graph obtained from G by deleting all the vertices in S and their incident edges. Similarly, the *deletion of an edge e* is the subgraph $G - e$. Then, for any set of edges X, $G - X$ is the graph obtained from G by deleting all the edges in X.

A (connected) *component* of a graph G is a connected subgraph H such that no subgraph of G that properly contains H is connected. Alternatively, a component is a maximal connected subgraph. A vertex v of G is a *cut-vertex* if $G - v$ has more components than G. A *block* of a graph is a maximally connected subgraph of G with no cut-vertex. Note that any two blocks of G have at most one vertex in common.

A *complete graph*, denoted by K_n, is a graph on n vertices with all the vertices being pairwise adjacent. A *clique* of a graph G is a maximal complete subgraph.

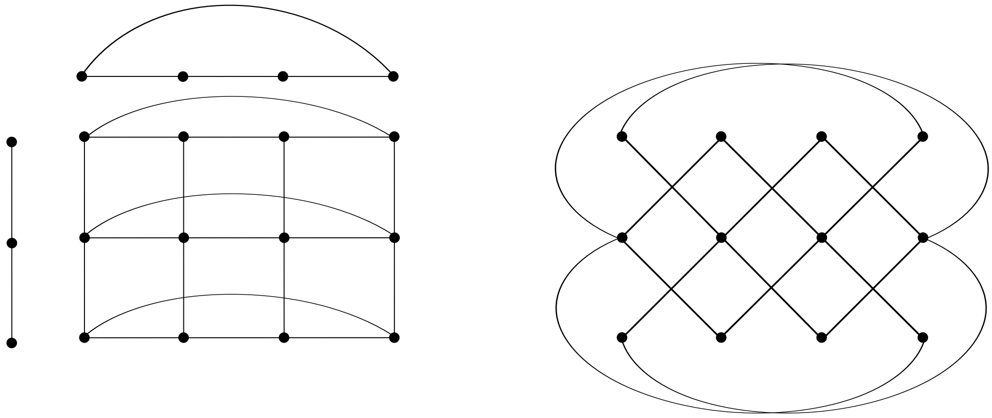
A *bipartite graph* is a graph whose vertex set can be partitioned into two non-empty subsets such that no two vertices in the same subset are connected by an edge. The two subsets are referred to as *partite sets*. Bipartite graphs are characterised by having no odd cycles. A *complete bipartite graph*, denoted by $K_{r,s}$, is a bipartite graph where each vertex from the partite set with r vertices is adjacent to all the vertices in the other partite set with s vertices. A *multipartite graph*, also referred to as a k-partite graph, consists of k disjoint non-empty sets of vertices such that vertices in the same set are not adjacent. A *complete k-partite graph*, denoted by $K_{r_1,r_2,\ldots r_k}$, is a k-partite graph whose partite sets have $r_1, r_2, \ldots, r_k$ vertices, respectively, such that every vertex in one partite set is adjacent to all the vertices in the other $k-1$ partite sets. If all the k partite sets have order r, then the graph is denoted by $K_{k(r)}$.

There are two operations that are important in topological graph theory, namely the subdivision of an edge and the contraction of an edge. The *subdivision of an edge $e = vw$* in a graph G replaces e by a new vertex u and adds two new edges vu and uw. A graph G is said to be a *subdivision of a graph H* either if $G = H$ or if G can be obtained from H by a sequence of edge subdivisions. Two graphs G and H are said to be *homeomorphic* if there is a graph $\mathcal{G}$ from which each of G and H can be obtained by a sequence (possibly empty) of subdivisions.

For an edge $e = uv$ of a graph G, the *contraction of the edge e* identifies its endpoints u and v by keeping the old adjacencies and removing the self-loop from $u = v$ to itself. When a graph H can be obtained from another graph G by a sequence of vertex-deletions, edge-deletions and edge-contractions, then H is a *minor* of G. Particularly, if H is a minor of G such that $H \neq G$, then H is a *proper minor* of G. Moreover, the following theorem shows that there is a relationship between subdivisions and minors of a graph.

Theorem 1.1.2. *If a graph G is a subdivision of a graph H, then H is a minor of G.*

Let G and H be two graphs with disjoint vertex-sets $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_k\}$ respectively. The *Cartesian product* of G and H, denoted by $G \Box H$, has vertex-set $V(G) \times V(H)$. Two vertices (v_i, w_i) and (v_j, w_j) are adjacent in $G \Box H$ if either $v_i = v_j$ and w_i is adjacent to w_j in H or $w_i = w_j$ and v_i is adjacent to v_j in G. The *direct product* of G and H, denoted by $G \times H$, has $V(G) \times V(H)$ as its vertex-set and two vertices (v_i, w_i) and (v_j, w_j) are adjacent in $G \times H$ if v_i is adjacent to v_j in G and w_i is adjacent to w_j in H. In literature, the direct product is also referred to as the Kronecker product, the tensor product, the categorical product and the conjunction of two graphs. In Figure 1.1 we give an example of the Cartesian product and direct product of the graphs C_4 and P_3.



$C_4 \square P_3$ $C_4 \times P_3$

Figure 1.1: The Cartesian Product $C_4 \square P_3$ (left) and the direct product $C_4 \times P_3$ (right)

1.2 | Introductory Concepts

The problem of embedding graphs on surfaces other than the sphere dates back to the "Map Colour Conjecture" due to Heawood in 1890 [14]. This conjecture, which is now a well-known result, establishes a formula for the number of colours required to draw a graph on a surface of genus g. The problem of determining the minimum genus of a graph has been proved to be substantially difficult. Thomassen in [24] showed that determining the minimum genus of a graph is NP-Hard, whilst determining whether an n vertex graph can be embedded on a surface of genus g is NP-Complete.

We now provide further definitions on graphs specifically related to surfaces and embeddings which have been adapted from [11, 12].

1.2.1 | Surfaces

In topological graph theory, the surfaces under consideration are closed, that is, compact and without a boundary. For this reason, the plane is not a surface since it is not compact. Surfaces can be of two types, orientable or non-orientable. The sphere S_0 is the first orientable surface from which other orientable surfaces are obtained. By cutting two holes in S_0 and attaching a handle from one hole to the other produces the *torus*, denoted by S_1. Repeating this process again yields the *double torus*, denoted by S_2. Thus, cutting $2g$ holes in S_0 and attaching g handles to the sphere S_0 in the manner described above yields the *orientable surface* S_g.

The *Möbius band* is obtained by giving a rectangle a half-twist and then pasting the right side to the left. Note that the Möbius band is not a surface since it has a boundary, hence, it is not closed. To obtain the first non-orientable surface, we cut a hole in a sphere and attach a Möbius band, also referred to as *crosscap*, along the boundary of the hole in the sphere. This surface is called the *projective plane* and is denoted by N_1. Adding two Möbius bands to a sphere yields the *Klein bottle*, denoted by N_2. In general, a *non-orientable surface* $N_{\bar{g}}$ is obtained by cutting $\bar{g}$ holes in the sphere and attaching a Möbius band to each one.

The *genus of an orientable surface* S_g is the number g of handles, whilst the *non-orientable genus of a non-orientable surface* $N_{\bar{g}}$ is the number $\bar{g}$ of crosscaps.

1.2.2 | Embeddings

A graph may be embedded on a surface other than the sphere, such as the surfaces mentioned above. In this case, an *embedding of a graph G on a surface S* is a continuous function $f : G \to S$ that maps G homeomorphically to its image $f(G)$. Intuitively, an embedding is a drawing of a graph on a surface such that no two edges cross each other. Whilst the plane is not a closed surface, it is topologically equivalent to the sphere S_0 with the point $(0, 0, 1)$ removed. Therefore, a graph can be embedded in the plane if and only if it can be embedded on the sphere. If a graph can be drawn in the plane without any two of its edges crossing, then it is referred to as a *planar graph*, whilst its image is the *plane graph*. The connected components of the complement of the image of an embedded graph are called *faces*. The *size* of a face is the number of edges bordering it. There is a well-known result, which was established by Euler for planar graphs, relating the number of vertices, edges and faces of a graph as stated in the theorem below.

Theorem 1.2.1. *(Euler's Formula) A connected, plane graph with n vertices, m edges and f faces satisfies the following,*

$$n + f - m = 2.$$

As a result, if the graph is disconnected, then Euler's Formula changes slightly.

Theorem 1.2.2. *A disconnected plane graph with n vertices, m edges, f faces and k components satisfies the following,*

$$n + f - m = 1 + k.$$

The next theorem is a fundamental graph theory result as it describes a necessary and sufficient condition for the non-planarity of a graph.

Theorem 1.2.3. *(Kuratowski's Theorem) [17] A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Such a characterisation is crucial when working in topological graph theory because it offers a clear and comprehensive criterion for identifying non-planar graphs. A similar result is credited to Wagner as a consequence of Theorem 1.1.2.

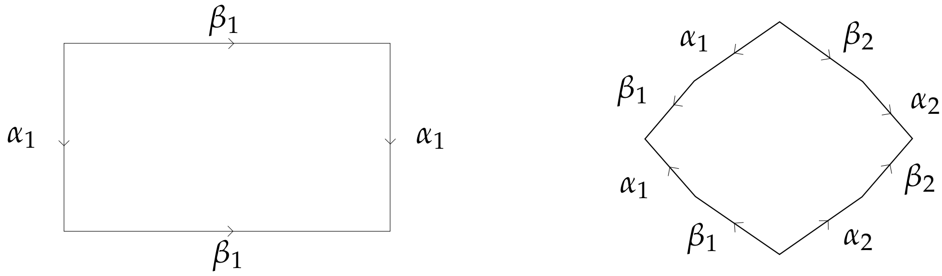
Theorem 1.2.4. *(Wagner's Theorem) A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor.*

An embedding of a graph is a 2-*cell embedding* if the interior of each face is homeomorphic to an open disk. The *orientable genus of a graph G*, denoted by $g(G)$ or g, is the smallest number of handles g of a surface S_g such that G can be embedded on S_g. In other words, a graph that can be embedded on an orientable surface of genus g but not on one of genus $g - 1$ is called a *graph of orientable genus g*, sometimes also referred to as the minimum genus. From now onwards, the orientable genus will be simply referred to as the genus. An embedding of a graph G of genus g on the surface S_g is called a *minimal embedding*. Youngs in [30] states that an embedding on a surface is a *maximal embedding* if no alternative embedding on the same surface has more components in the complement of the embedded graph. He proves that an embedding of a graph G is minimal if and only if it is a maximal 2-cell embedding. Thus, since in this dissertation we are concerned with minimal embeddings, we shall only be considering 2-cell embeddings.

Similarly, a graph that can be embedded without crossings on a non-orientable surface of genus $\bar{g}$ but not on one of genus $\bar{g} - 1$ is called a *graph of non-orientable genus* $\bar{g}$, sometimes also referred to as the minimum non-orientable genus. The *minimum non-orientable genus embedding* of a graph G is an embedding of G into a closed surface of minimum non-orientable genus.

In this study, only orientable surfaces and the orientable genus of a graph will be considered.

When the number of handles added to the sphere is relatively small, it may be useful to consider the surface flattened into a polygon. For an orientable surface S_g, take a $4g$-gon and identify its sides according to the pattern $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \ldots, \alpha_g, \beta_g, \alpha$ Figure 1.2 illustrates the flattenings for the torus and double-torus.



(a) (b)

Figure 1.2: (a): Flattening for the torus, (b): Flattening for the double-torus

The Handshaking Lemma, found in Lemma 1.1.1, can be described in terms of the sizes of faces in an embedding of a graph rather than the degrees of the vertices.

Theorem 1.2.5. *For a graph G with m edges and with a 2-cell embedding on a surface*

S_g with k faces labelled $f_1, f_2, \ldots, f_k$, the following holds,

$$2m = \sum_{i=1}^{k} |f_i|,$$

where $|f_i|$ denotes the size of the face f_i.

Euler's Formula, found in Theorem 1.2.1, is generalised to graphs that are embedded on surfaces other than the plane.

Theorem 1.2.6. *(Generalised Euler's Formula) Let G be a connected graph with n vertices and m edges. Consider a 2-cell embedding of G on an orientable surface S_g with genus g, resulting in f faces. Then,*

$$n + f - m = 2 - 2g.$$

The following result was proved by J.W.T Youngs in [30].

Theorem 1.2.7. *[30] Every embedding of a connected graph G of genus g on S_g, where $g \geq 0$, is a 2-cell embedding.*

Using Theorem 1.2.6 and Theorem 1.2.7, we are in a position to state the following result.

Corollary 1.2.8. *Let G be a connected graph with n vertices and m edges that is embedded on an orientable surface of genus $g(G)$, resulting in f faces. Then,*

$$n + f - m = 2 - 2g(G).$$

Such a result is highly useful when we need to establish the genus of a graph after establishing the number of faces in an embedding of G on S_g. If the sizes of the faces of the 2-cell embedding of the graph in consideration are known, then, Theorem 1.2.5 is applied to determine the number of faces. For the case when each face has size three, the embedding is referred to as a *triangular embedding*, whilst when each face has size four, the embedding is called a *quadrilateral embedding*.

Note that a disconnected graph has no 2-cell embedding and for an embedding of a graph G on a surface S_g to be a 2-cell embedding, every handle must carry at least an edge of G. The next lemma is an application of the Generalised Euler's Formula for the case when a graph has minimum degree at least three.

Lemma 1.2.9. *[29] Let G be a connected graph having minimum degree at least three, which is minimally embedded on an orientable surface of genus g. Let f be the number of faces of this embedding, f_i and n_i be the number of i-sided faces and vertices of degree i, respectively. Then,*

$$g(G) = 1 - \frac{m}{6} + \frac{1}{6}\sum_{i\geq 3}(i-3)(f_i + n_i).$$

Proof. We apply Generalised Euler's Formula found in Theorem 1.2.6. Since G has minimum degree at least three, then each face is bounded by at least three edges. Note that $f = \sum_{i\geq 3} f_i$, $n = \sum_{i\geq 3} n_i$ and $2m = \sum_{i\geq 3} in_i = \sum_{i\geq 3} if_i$. Hence, multiplying the equation Corollary 1.2.8 by three and substituting the above relations, we get:

$$\begin{aligned}
6g(G) &= 6 - 3n - 3f + 3m \\
&= 6 - 3\sum_{i\geq 3} n_i - 3\sum_{i\geq 3} f_i + 4m - m \\
&= 6 - 3\sum_{i\geq 3}(n_i + f_i) + \sum_{i\geq 3}(in_i + if_i) - m \\
&= 6 - \sum_{i\geq 3}(i-3)(n_i + f_i) - m
\end{aligned}$$

Hence, the result follows. □

Another well-known result establishes the genus of graphs with k blocks, as seen in the next theorem. The proof is quite technical and hence it is being ommitted here.

Theorem 1.2.10. *[15] If G is a connected graph having having k blocks, $B_1, \ldots, B_k$, then $g(G) = \sum_{i=1}^{k} g(B_i)$*

If G is not connected, then suppose it has k components. We can form a connected graph H from G such that H has one new vertex which is connected to a vertex from each of the k components from G. The blocks of H are the components of G along with these k edges, however, the insertion of these new edges does not affect the genus. As a consequence, we have the following two corollaries about the genus of a graph with several components.

Corollary 1.2.11. *[15] The genus of a graph (connected or not) is the sum of the genera of its blocks.*

Corollary 1.2.12. *[15] The genus of a graph is the sum of the genera of its components.*

Therefore, even though a disconnected graph cannot have a 2-cell embedding, its components can be embedded on separate surfaces and the genus of the graph is the sum of the genera of the surfaces the components are embedded on.

1.2.3 | Crossing Numbers

Let G be a graph. Then a *good drawing* of G is one in which:

- No edge crosses itself.
- Any two edges have at most one point in common, which can either be a crossing or a vertex.
- No three edges have a common point other than a vertex.

An *optimal drawing* of G is a good drawing exhibiting the least number of crossings. The *crossing number* of G, denoted by $\nu(G)$, is the number of crossings in an optimal drawing. The genus of a graph and the crossing number of a graph are related to each other as follows.

Theorem 1.2.13. *For a graph G, the genus of G is at most its crossing number, that is*

$$g(G) \leq \nu(G).$$

Intuitively, we start from an optimal drawing of the graph G on the plane and for each crossing, a handle is introduced. However, a handle might be able to resolve more than one crossing, hence, the inequality is established. We note that $\nu(G) = 0$ if and only if G is planar and $\nu(K_5) = \nu(K_{3,3}) = 1$.

Since inserting vertices of degree two does not affect the crossing number, the next result follows immediately.

Theorem 1.2.14. *If two graphs G and H are homeomorphic, then $\nu(G) = \nu(H)$.*

1.3 | Overview

In Chapter 2 we shall establish the genus of one of the most popular bipartite graphs, namely, the complete bipartite graph $K_{p,q}$. Here, we introduce current graphs, that serve as the main tool in Ringel's proof that established the genus of $K_{p,q}$.

In Chapter 3, we look at general results on the Cartesian product of bipartite graphs. Then, we establish the genus of several families of the Cartesian product of bipartite graphs involving cycles, paths, the complete graph $K_{2r,2r}$ and their repeated Cartesian products.

In Chapter 4, we consider another type of graph product, namely the direct product of two graphs. This is then extended to the repeated direct product of graphs, and the genus of some families of repeated direct product graphs is established, involving cycles, paths, the j-cube and the complete bipartite graph.

We conclude in Chapter 5 by reviewing the most important results of this study. We present some results concerning the genus of other families of graphs, including other products of graphs, and put forward two open problems.

We note that some results in Chapter 3 are new and have been submitted to be considered for eventual publication in a peer-reviewed journal [9] (refer to Appendix). In particular, these include Theorem 3.2.8, Theorem 3.2.10, Corollary 3.2.11, Theorem 3.5.7, Theorem 3.5.8, Theorem 3.5.9 and Theorem 3.5.10. Moreover, another original result establishing the genus of $K_{1,3} \square P_{r+1}$, found in Lemma 3.6.2, was proved in the course of obtaining the crossing number of $K_{1,3} \square P_{r+1}$ on any surface S. The corresponding paper is in its final writing stage. We also note that we give original proofs to some known results, including Proposition 3.1.6, Lemma 3.4.1, Lemma 3.5.3 and Lemma 4.1.1.

2

The Genus of $K_{p,q}$

This chapter takes into consideration the genus of a well-known family of graphs, namely the complete bipartite graphs. This result was first proved by Ringel in 1965 [21], and was later proved from a different perspective by Bouchet [3]. The following is the formula used to compute the genus of a complete bipartite graph $K_{p,q}$.

Theorem 2.0.1. *[3, 21] The genus of the complete bipartite graph $K_{p,q}$ is given by*

$$g(K_{p,q}) = \left\lceil \frac{(p-2)(q-2)}{4} \right\rceil .$$

In the subsequent section, we delve into a more specific category of complete bipartite graphs, specifically, those characterized by partite sets of equal and even size.

2.1 | Quadrilateral Embeddings

We now consider an instance of a complete bipartite graph where the partite sets are of equal and even size.

Example 2.1.1. *Consider the graph $K_{4,4}$ and call its partite sets A and B. Label the vertices in A and B such that $A = \{1,2,3,4\}$ and $B = \{5,6,7,8\}$. Define the following* *permutations on the vertices of $K_{4,4}$:*

$$
\begin{aligned}
P_1 = P_3 &= (5\ 6\ 7\ 8),\\
P_2 = P_4 &= (8\ 7\ 6\ 5),\\
P_5 = P_7 &= (1\ 2\ 3\ 4),\\
P_6 = P_8 &= (4\ 3\ 2\ 1).
\end{aligned}
\tag{2.1}
$$

Let $W = \{(a,b) : \{a,b\} \in E(K_{4,4})\}$ be the set of ordered pairs on the vertices of $K_{4,4}$ and let $P : W \to W$ be defined by $P((a,b)) = (b, P_b(a))$. Using (2.1), the orbits formed under P are given by:

$$
\begin{aligned}
S_1 &= \{(1,5),(5,2),(2,8),(8,1)\},\\
S_2 &= \{(1,6),(6,4),(4,5),(5,1)\},\\
S_3 &= \{(1,7),(7,2),(2,6),(6,1)\},\\
S_4 &= \{(1,8),(8,4),(4,7),(7,1)\},\\
S_5 &= \{(2,5),(5,3),(3,6),(6,2)\},\\
S_6 &= \{(2,7),(7,3),(3,8),(8,2)\},\\
S_7 &= \{(3,5),(5,4),(4,8),(8,3)\},\\
S_8 &= \{(3,7),(7,4),(4,6),(6,3)\}.
\end{aligned}
\tag{2.2}
$$

Let the class C be the collection of sets of orbits under P, that is, the elements of C are the sets in (2.2). For $i = 1, \ldots, 8$, the class C has the following properties:

1. *each element of S_i occurs in exactly two sets of C,*
2. *no proper subclass of C has property (1) and*
3. *each S_i is cyclically ordered.*

It is a result of Edmonds [6] that when a class of sets has the properties mentioned in Example 2.1.1, that class is a representation for a topologically uniquely oriented polyhedron.

Theorem 2.1.2. *[6] A class of sets with the following properties:*

1. *every element of these sets occurs exactly twice,*

2. *no proper subclass of the class has property (1), that is, every element occurring once in a set of the subclass does not occur twice in sets of the subclass,*

3. *the sets of the class are cyclically ordered*

is a representation for a topologically uniquely oriented polyhedral surface (polyhedron) where the sets of the class correspond to its vertices and the distinct elements correspond to its edges. Every oriented polyhedral surface has such a representation.

The proof of this theorem is omitted as it is goes beyond the scope of our study.

We now derive a formula for the genus of any bipartite graph which has a quadrilateral embedding.

Proposition 2.1.3. *If the bipartite graph G with n vertices and m edges has a quadrilateral embedding, then that embedding is minimal and $g(G) = 1 + \frac{m}{4} - \frac{n}{2}$.*

Proof. Since G is bipartite, the smallest possible number of edges that can border a face is four, implying that a quadrilateral embedding of G is minimal. Therefore, by double-counting, $4f = 2m$. Substituting this value in Corollary 1.2.8, we get

$$\begin{aligned} 2g(G) &= 2 + m - n - f \\ \Rightarrow 2g(G) &= 2 + m - n - \frac{m}{2} \\ \Rightarrow g(G) &= 1 + \frac{m}{4} - \frac{n}{2} \end{aligned}$$

Therefore, the result follows. □

From Example 2.1.1, each set given in (2.2) corresponds to a quadrilateral face of $K_{4,4}$, totalling to eight quadrilateral faces. These can be partitioned into four subsets of two faces each as follows:

$$\begin{aligned} A_1 &= \{S_1, S_8\} \\ A_2 &= \{S_2, S_6\} \\ A_3 &= \{S_3, S_7\} \\ A_4 &= \{S_4, S_5\} \end{aligned} \tag{2.3}$$

Note that each set in (2.3) contains all eight vertices of $K_{4,4}$. The approach used above for $K_{4,4}$ can be generalised for $K_{2r,2r}, r \geq 2$ as shown in Remark 2.1.4 and Lemma 2.1.5.

Remark 2.1.4. *We first describe a quadrilateral embedding for $K_{2r,2r}$ given by Ringel [21] (also found in [28, 30]) that yields the required value of the genus. Label the vertex set of $K_{2r,2r}$ by $V(K_{2r,2r}) = \{1, \ldots, 4r\}$, with the adjacencies at vertex i given by*

$$N_{K_{2r,2r}}(i) = \begin{cases} \{j : 2m+1 \leq j \leq 4r\}, & \text{for } 1 \leq i \leq 2r, \\ \{j : 1 \leq j \leq 2r\}, & \text{for } 2r+1 \leq i \leq 4r. \end{cases}$$

Define the cyclic permutations $P_i : N_{K_{2r,2r}}(i) \to N_{K_{2r,2r}}(i)$, for $i = 1, \ldots, 4r$ by:

$$\begin{aligned} P_1, P_3, \ldots, P_{2r-1} &: (2r+1\ 2r+2\ \ldots\ 4r), \\ P_2, P_4, \ldots, P_{2r} &: (4r\ 4r-1\ \ldots\ 2r+1), \\ P_{2r+1}, P_{2r+3}, \ldots, P_{4r-1} &: (1\ 2\ \ldots\ 2r), \\ P_{2r+2}, P_{2r+4}, \ldots, P_{2r} &: (2r\ 2r-1\ \ldots\ 1). \end{aligned} \tag{2.4}$$

By Theorem 2.1.2, the collection of permutations described above uniquely determines a 2-cell embedding of $K_{2r,2r}$ on an orientable surface, once an orientation on the edges is chosen. Therefore, let (a, b) represent the directed edge from vertex a to vertex b, corresponding to the edge $\{a, b\}$ in $E(K_{2r,2r})$. Define the set of all directed edges of $K_{2r,2r}$ as $W = \{(a, b) : \{a, b\} \in E(K_{2r,2r})\}$ and let $P : W \to W$ be defined by $P((a, b)) = (b, P_b(a))$. Then, the orbits under P correspond to 2-cell faces of the embedding. Note that for $r \geq 2$, the number of quadrilateral faces is $2r^2$ because every pair of edges incident to a vertex is in a face of $K_{2r,2r}$. Thus, each vertex is in r faces and since one partite set covers all the faces in the embedding, there are $2r.r = 2r^2$ faces. Therefore, by Proposition 2.1.3 we get that:

$$\begin{aligned} g(K_{2r,2r}) &= 1 + \frac{4r^2}{4} - \frac{4r}{2} \\ \Rightarrow g(K_{2r,2r}) &= 1 + r^2 - 2r \\ \Rightarrow g(K_{2r,2r}) &= (r-1)^2. \end{aligned}$$

This coincides with the value of the genus given in Theorem 2.0.1. We note that the case when $r = 1$ produces only one face and hence the above construction cannot be used. In *this case, we have that* $K_{2,2} = C_4$*, therefore,* $g(C_4) = 0$*, which conforms to the result above.*

The following lemma shows that there is a partition of the quadrilateral faces for $K_{2r,2r}$ such that each subset of the partition covers all the vertices of the graph.

Lemma 2.1.5. *[28] For the embedding of* $K_{2r,2r}$*, for* $r \geq 2$ *given in Remark 2.1.4, the* $2r^2$ *quadrilateral faces can be partitioned into* $2r$ *subsets of* r *faces each, such that each subset contains all* $4r$ *vertices of the graph.*

Proof. The orbits under the permutation P as defined above using permutations P_i, for $1 \leq i \leq 4r$, are given by:

$$\begin{aligned} &\{2g-1, 2h-1, 2g, 2h-2 : 1 \leq g \leq r, r+1 < h \leq 2r\}, \\ &\{2g-1, 2h-1, 2g, 4r : 1 \leq g \leq r, h = r+1\}, \\ &\{2j, 2k-1, 2j+1, 2k : r+1 \leq k \leq 2r, 1 \leq j < r\}, \\ &\{2j, 2k-1, 1, 2k : r+1 \leq k \leq 2r, j = r\}. \end{aligned} \tag{2.5}$$

Since an orbit corresponds to a quadrilateral face, the number of faces in a quadrilateral embedding is $2r^2$. This follows from (2.5) since the first and third sets generate $r(2r - r - 1) = r^2 - r$ faces each whilst the second and fourth sets generate r faces each, thus, summing them up gives $2r^2$. Now, these faces are partitioned into $2r$ subsets as follows. Fix i, the r faces in subset $2i - 1$ are established by selecting $h = r + g + i$, with $1 \leq g \leq r$, where in $(g + i)$ modulo r, r is written instead of 0. The r faces of subset $2i$ are established by selecting $k = m + j + i$, for $1 \leq j \leq r$, again taking $(j + i)$ modulo r, where r is written instead of 0. As i runs from 1 to r, we obtain $2r$ sets in total, each having r faces. The sets are mutually disjoint by the way they were chosen from the orbits above. Also, each set containing r faces has all $4r$ vertices of the graph. $\square$

Remark 2.1.6. *When* $r = 1$ *in Lemma 2.1.5, the two orbits obtained are* $\{1, 3, 2, 4\}$ *and* $\{2, 3, 1, 4\}$*, which describe the same face. This is a contradiction to the statement of Lemma 2.1.5, as the orbits should be mutually disjoint, that is, they describe different faces.*

Refer to Example 2.1.1 for an instance of the construction given in Remark 2.1.4, where the cyclic permutations from (2.4) are given by (2.1) and the orbits in (2.5) are given by the unique elements of the sets in (2.2).

2.2 | The Proof by G. Ringel

The problem of finding the genus of the complete bipartite graph is divided into several sub-problems, arising from the congruence classes of the formula in Theorem 2.0.1. Ringel was the first to establish the genus by using current graphs to describe an embedding for some values of p and q. We first give a brief description of current graphs, along with some examples and preliminary results. These findings will then be employed to establish the genus of the complete bipartite graph.

2.2.1 | The Face-Tracing Algorithm

The definitions in this section are taken from 'A Handbook of Graph Theory' edited by Gross, Yellen and Zhang [12]. Whilst so far we have considered a graph to be the pair (V, E), it can also be seen as a combinatorial structure $\langle V, E \rangle$ together with a ground set S as follows:

- The elements in the set S are referred to as *half-edges*,
- E is a partition of S into pairs, such that each half-edge is paired with its counterpart, that is, the other half of the same edge,
- V is a partition of the half-edges according to the vertex at which they are incident.

The partition E can be seen as the set of orbits of an involution τ. Then, a *rotation at a vertex v* is an ordered list of the half-edges incident to v. A half-edge can only be assigned one of two orientation types, one of which is referred to as the plus direction whilst the other is referred to as the reverse direction. The *surface rotation at a vertex v* of a graph embedding is the cyclic ordering of the half-edges at v on the surface. If this surface is orientable, this ordering is taken to be consistent with the orientation.

A *global rotation*, also called a rotation system, on a graph is an assignment of a rotation to each vertex. Thus, we have a permutation ρ on the set of half-edges whose orbits are the rotations at the vertices. The *global surface rotation of an embedded graph* is the set of surface rotations at all the vertices. The *induced embedding* of a global rotation ρ on a graph G is an embedding of G whose global surface rotation is ρ.

A *face tracing* for a global rotation on a graph is a list of the boundary walks of the faces of an induced embedding. The *signature of a graph* $G = (V, E)$ is a subset Λ of E whose edges are called switches. They represent the edges whose traversal switches the sense of orientation in an embedding. Therefore, a *generalised rotation* is a pair (ρ, Λ) of a global rotation and a signature.

The Face-Tracing algorithm is used to produce a list of all the face-boundaries of an induced embedding, as seen in Algorithm 1.

Algorithm 1: The Face-Tracing Algorithm [12]

```
Input: List of half-edges E, involution τ, rotation ρ
Output: A list of face-boundaries of an induced embedding
Initialisation: Mark all the half-edges in E as unused
while any unused half-edges remain do
    Choose the next unused half-edge y from E
    Start a new cycle by writing "("
    x := y
    while x ≠ y do
        Write x in the current cycle
        x := ρ(τ(x))                    // Choose the next half-edge
    end
    Close the current cycle by writing ")"
end
```

We note that the rotation ρ given as input is a global rotation, which is different from a generalised rotation. In the latter case, the algorithm found in [11] is a modification of Algorithm 1 as it accounts for those edges that switch the orientation in an embedding.

2.2.2 | A Brief Introduction to Current Graphs

By $G \to S$, we denote the embedding of a graph G on an orientable surface S. Let G be a directed graph, with $E(G)$ denoting the set of directed edges, where edges with a plus direction are denoted by e^+ and edges with the reverse direction are denoted by e^-. Let $\mathcal{B}$ be a finite group. Then, an *ordinary current assignment* is a function $\beta : E(G) \to \mathcal{B}$ from the set of directed edges of G to the finite group $\mathcal{B}$ such that for every directed edge $e \in E(G)$, $\beta(e^-) = (\beta(e^+))^{-1}$. The values assigned by β are called *currents* whilst $\mathcal{B}$ is referred to as the *current group*. Therefore, a *current graph* is the pair $(G \to S, \beta)$.

Let $F(G)$ denote the set of faces of a current graph $(G \to S, \beta)$. A *derived graph* G_β has vertex set $F(G) \times \mathcal{B}$ and edge set $E(G) \times \mathcal{B}$. Therefore, the faces of the current graph correspond to the vertices of the derived graph. For $e \in E(G)$, $f \in F(G)$ and $b \in \mathcal{B}$, the vertices and derived edges of a derived graph G_β are denoted by (f, b) and (e, b) respectively. A derived edge (e, b) is given the plus direction if it has source vertex (f, b) and terminal vertex (g, bc), where $\beta(e^+) = c$ and the rotation at the initial vertex of the edge e^+ carries face f to face g. Such an edge is denoted by $(e, b)^+$ or by (e^+, b). Similarly, an edge with a reverse direction, denoted by $(e, b)^-$, goes from vertex (g, bc) to (f, b). However, note that the derived edge (e^-, b) has source vertex (g, b) and terminal vertex (f, bc^{-1}).

Remark 2.2.1. *The rotation system for G_β is obtained from $G \to S$ as follows. If the boundary walk of a face f is given by $e_1^{\epsilon_1} e_2^{\epsilon_2} \dots e_n^{\epsilon_n}$, then, the rotation at vertex (f, b) is given by $(e_1, b_1)^{\epsilon_1}(e_2, b_2)^{\epsilon_2} \dots (e_n, b_n)^{\epsilon_n}$, where $b_i = b$ if $\epsilon_i = +$ and $b_i = b\beta(e_i^-)$ if $\epsilon_i = -$. For brevity, the rotation may be denoted by $(e_1^{\epsilon_1}, b)(e_2^{\epsilon_2}, b) \dots (e_n^{\epsilon_n}, b)$. Note that the rotation at a vertex (f, ab) is simply the rotation (f, b) with the second coordinates multiplied on the left by a.*

The following is a result by Gross and Alpert that establishes the number of faces in an embedding of a derived graph.

Theorem 2.2.2. *[10] Let $e_1^{\epsilon_1}, e_2^{\epsilon_2}, \dots e_n^{\epsilon_n}$ be the rotation at vertex v of the current graph $(G \to S, \beta)$ and let c_i be the current carried by the edge e_i that has v as its initial vertex. Let $c = c_i \dots c_n$, and let r be the order of c in $\mathcal{B}$. Then the derived embedding has $\frac{|\mathcal{B}|}{r}$*

faces corresponding to vertex v, each of size rn and each of the form

$$(e_1^{\epsilon_1}, b), (e_2^{\epsilon_1}, bc_1), (e_3^{\epsilon_1}, bc_1c_2), \ldots, (e_n^{\epsilon_1}, bc_1c_2 \ldots c_{n-1})$$
$$(e_1^{\epsilon_1}, bc), (e_2^{\epsilon_1}, bcc_1), (e_3^{\epsilon_1}, bcc_1c_2), \ldots, (e_n^{\epsilon_1}, bcc_1c_2 \ldots c_{n-1})$$
$$\vdots$$
$$(e_1^{\epsilon_1}, bc^r) = (e_1^{\epsilon_1}, b).$$

An important law in the theory of current graphs is the Kirchoff Current Law (KCL), which states that the sum of currents in an Abelian group is the group identity. When the group is non-Abelian, the product of currents is used instead. Note that the product $c_1 \ldots c_n$ in Theorem 2.2.2 is referred to as the excess current at a vertex v. Hence, the KCL holds at a vertex v if the excess current is the identity of $\mathcal{B}$.

Example 2.2.3. *Consider the current graph $(G \to S_0, \beta)$, with current group $\mathbb{Z}_5$, given in Figure 2.1. Let e_i^+ denote the edge (v_i, v_{i+1}), for $i \in \{1, 2, 3\}$. Here, the graph*

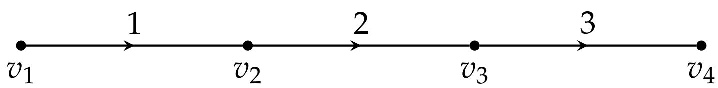


Figure 2.1: Current Graph G

is planar and the embedding $G \to S_0$ only has one face, denoted by f, whose boundary walk for f is given by $e_1^+ e_2^+ e_3^+ e_3^- e_2^- e_1^-$.

By definition, the vertex set of the derived graph, G_β, is $\{(f, 0), (f, 1), (f, 2), (f, 3), (f, 4)\}$. By Remark 2.2.1, the rotation at vertex $(f, 0)$ is $(e_1, 0)^+(e_2, 0)^+(e_3, 0)^+(e_3, 2)^-(e_2, 3)^-(e_1, 4)^-$. For example, the derived edge $(e_3, 2)^-$ corresponds to the edge e_3^- from the boundary walk of f, where $2 = 0 + \beta(e_3^-) = (\beta(e_3))^{-1}$, by Remark 2.2.1 and by the definition of the ordinary current assignment. These edges are all adjacent to $(f, 0)$ in G_β, however, two of the edges listed turn out to be repeated. Consider the edges $(e_2, 0)^+$ and $(e_3, 2)^-$. The initial vertex of $(e_2, 0)^+$ and $(e_3, 2)^-$ is $(f, 0)$ whilst their terminal vertex is $(f, 2)$. Note that, by definition, $(e_3, 2)^-$ has terminal vertex $(f, 2)$ and initial vertex $(f, (2 + \beta(e_3^+)))$ which reduces to $(f, 0)$ since addition is under modulo 5. The derived edges $(e_3, 0)^+$ and $(e_2, 3)^-$ also represent the same edge, hence, the rotation at $(f, 0)$ has exactly four unique edges. The rotation at $(f, 0)$ is represented pictorially in Figure 2.2.

This process is repeated for all the other four vertices, which would result in the complete graph K_5.

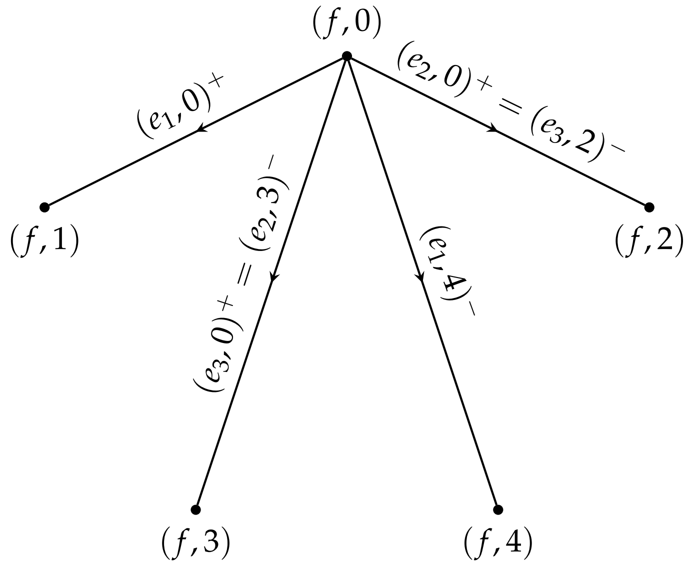


Figure 2.2: Rotation at vertex $(f,0)$ in G_β

2.2.3 | An Application of Current Graphs

We employ Theorem 2.2.2 to derive a quadrilateral embedding of $K_{p,q}$, for the case when p and q are both even.

Lemma 2.2.4. *[11] The complete bipartite graph $K_{p,q}$, where p and q are both even integers, has a quadrilateral embedding.*

Proof. Consider the current graph G on p vertices given in Figure 2.3, with currents given from the cyclic group $\mathbb{Z}_q$. All the edges in G are assigned the current one, with the directions of the edges alternating between the positive and reverse orientation.

By applying the Face Tracing Algorithm to G, we note that there are only two faces generated by this current graph, denoted by f and g. Let e_i denote the edge connecting vertex v_i to v_{i+1}. The boundary walk for f is given by $e_1e_2^-e_3e_4^-e_5e_6^- \ldots e_p^-$ whilst the one for g is given by $e_6e_5^-e_4e_3^-e_2e_1^-e_p \ldots e_7^-$. We

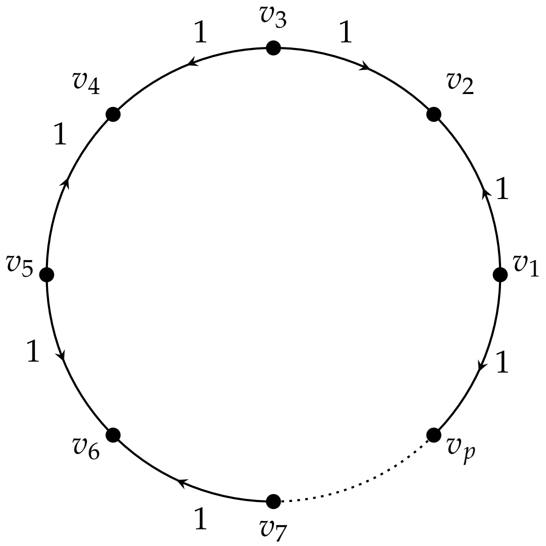


Figure 2.3: Current Graph on p vertices with currents in $\mathbb{Z}_q$ [11]

assign a rotation to each vertex in G in such a way that the edge listed first occurs in face f, and the edge following it occurs in face g.

The vertex-set of the derived graph G_β is $\{(f,i) : i \in \mathbb{Z}_q\} \cup \{(g,i) : i \in \mathbb{Z}\}$. Consider the vertex $(f,0)$ in $V(G_\beta)$. By Remark 2.2.1, the rotation at $(f,0)$ is given by

$$(e_1,0)^+(e_2,q-1)^-(e_3,0)^+(e_4,q-1)^- \ldots (e_p,q-1)^-.$$

The first edge in this rotation has source vertex $(f,0)$ and terminal vertex $(g,1)$ as the rotation at v_1 carries face f to face g. The edge $(e_2,q-1)^-$ has source vertex $(g,0)$ and terminal vertex $(f,q-1)$. The rest of the edges in the rotation at $(f,0)$ are repeated since edges of the form $(e_i,0)^+$ will always have source vertex $(f,0)$ and terminal vertex $(g,1)$ by the way rotations on the vertices of G were fixed and due to the current assigned to an edge always being one. Furthermore, edges of the form $(e_i,q-1)^-$ always have source vertex $(g,0)$ and terminal vertex $(f,q-1)$ by definition of edges in a derived graph. Extending this to other vertices in $V(G_\beta)$, we note that there is an edge between (f,i) and $(g,i+1)$ for all $0 \le i \le q-1$ and there are no other edges. Therefore, the derived graph has two components, where the vertex-set of one component is $\{(f,i) : i \text{ even}\} \cup \{(g,j) : j \text{ odd}\}$. Each vertex v_k of the current graph has excess current ± 2 and thus has order $\frac{q}{2}$ in $\mathbb{Z}_q$. By Theorem 2.2.2, the derived embedding has two faces corresponding to v_k, and since v_k has exactly two neighbours, each face is bordered by q edges. We note that one face traverses the vertices form one component, whilst the other face passes through the vertices in the other component. Thus, $2p$ faces are present in the derived graph, all of which being q-gons.

From faces corresponding to one component, we add a vertex w inside each q-gon and add edges joining w to the q original vertices. Then, we delete all the original edges. The resulting graph is the complete bipartite graph $K_{p,q}$ and every face of the resulting embedding is a quadrilateral. □

A similar argument is used to construct a quadrilateral embedding for the complete bipartite graph when p is odd and $q \equiv 2 \bmod 4$.

Lemma 2.2.5. *[11] The complete bipartite graph $K_{p,q}$, where p is an odd integer and $q \equiv 2 \bmod 4$, has a quadrilateral embedding.*

Proof. Consider the current graph G given in Figure 2.4. G is on p vertices and currents are assigned from the group $\mathbb{Z}_q$. Let e_i denote the edge connecting vertex v_i to v_{i+1}. The orientation of the edges is alternating between positive and reverse, however, since p is odd, the edge between v_{p-1} and v_p is given the same orientation as the edge (v_{p-1}, v_{p-2}) as illustrated in Figure 2.4. All edges are assigned current 1, except for edge (v_p, v_{p-1}) which is assigned current 3. From the Face Tracing Algorithm, the current graph generates two faces, f and g, whose boundary walks are given by $e_1^- e_2^+ e_3^- \dots e_{p-2}^- e_{p-1}^- e_p^+$ and $e_1^+ e_p^- e_{p-1}^+ e_{p-2}^+ \dots e_3^+ e_2^-$ respectively.

All vertices in this current graph have excess current ± 2, therefore, they have order $\frac{q}{2}$. By Theorem 2.2.2, all vertices generate two faces of size q. In each case, there is a face that passes through one component and another face that passes through the vertices of the other component. Thus, we place a vertex w inside each face that passes through the first component, we add edges joining that vertex to the original vertices then we delete the original edges. This results in the complete bipartite graph $K_{p,q}$ with all the faces being quadrilateral.

□

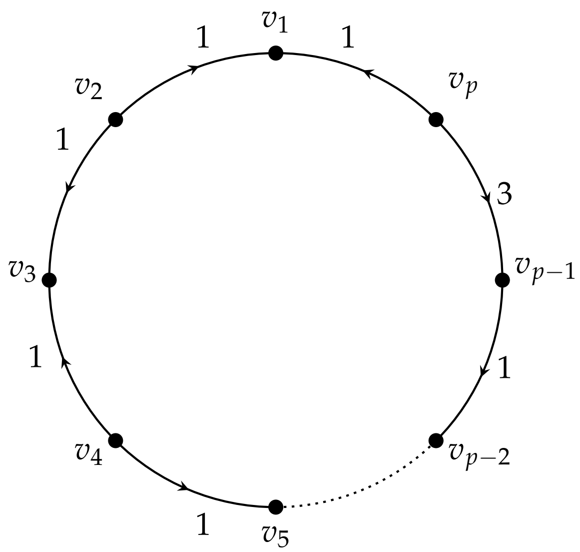


Figure 2.4: Current Graph on p vertices with currents in $\mathbb{Z}_q$, with p odd [11]

For the remaining cases, constructions are given to construct an embedding with the required genus.

Theorem 2.2.6. *[11, 21] The genus of the complete bipartite graph $K_{p,q}$, where p and q are positive integers, is given by*

$$g(K_{p,q}) = \left\lceil \frac{(p-2)(q-2)}{4} \right\rceil.$$

Proof. The quadrilateral embeddings for the cases when p and q are both even and when p is odd and $q \equiv 2 \bmod 4$ are given by Lemma 2.2.4 and Lemma 2.2.5, respectively. By Proposition 2.1.3, the genus of $K_{p,q}$ in both of these cases is given by

$$\begin{aligned} g(K_{p,q}) &= 1 + \frac{pq}{4} - \frac{p+q}{2} \\ &= \frac{4 + pq - 2q - 2p}{4} \\ &= \frac{(q-2)(p-2)}{4}. \end{aligned}$$

Consider the case when $p \equiv 3 \bmod 4$ and $q \equiv 1 \bmod 4$. Then $K_{p-1,q}$ has a quadrilateral embedding since q is odd and $p - 1 \equiv 2 \bmod 4$. In such an embedding, there is a vertex v in the partite set of size $p-1$ forming part of q quadrilateral faces, as shown in figure Figure 2.5.

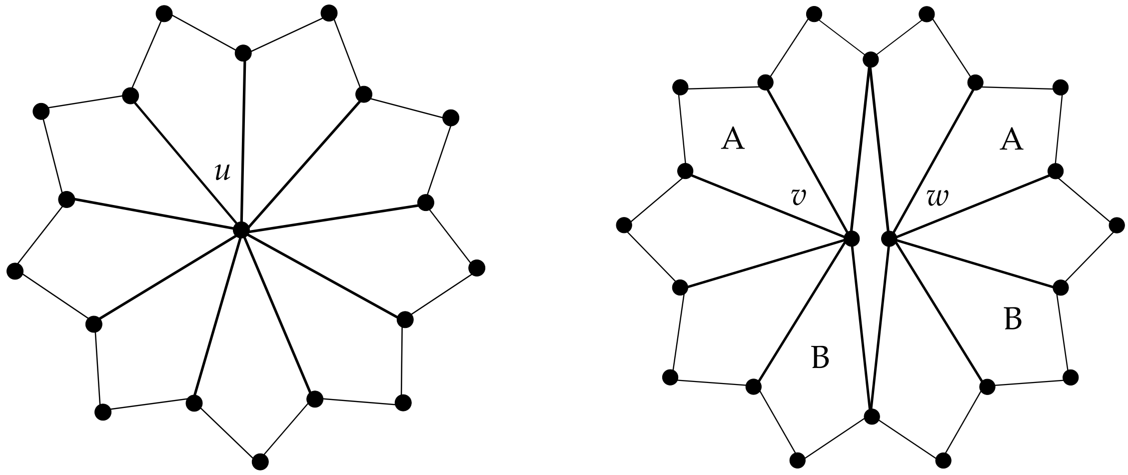


Figure 2.5: Splitting a vertex u in $K_{p-1,q}$

The vertex u is split in two vertices, v and w, and both are connected to the nearest half of the surrounding q vertices, as illustrated in Figure 2.5. To complete the embedding of $K_{p,q}$, there are some edges left to connect. Note that the embedding for $K_{p-1,q}$ has genus $\frac{(p-3)(q-2)}{4}$, whilst the desired embedding for $K_{p,q}$ has genus $\frac{(p-2)(q-2)+1}{4}$, for $p \equiv 3 \bmod 4$ and $q \equiv 1 \bmod 4$. Writing $p = 4r+3$ is the same as $p \equiv 3 \bmod 4$ and similarly, $q \equiv 1 \bmod 4$ can be written as $q = 4t+1$. Therefore, substituting the equations for p and q into the expression $\left\lceil \frac{(p-2)(q-2)}{4} \right\rceil$ yields $\left\lceil \frac{4(4tr+t-r)-1}{4} \right\rceil$, whose ceiling is $\frac{4(4tr+t-r)-1+1}{4} = \frac{(p-2)(q-2)+1}{4}$. The difference between $g(K_{p-1,q})$ and $g(K_{p,q})$ is $\frac{q-1}{4}$, thus at most that amount of handles can be employed to insert the extra edges at v and w. From the faces corresponding to vertex u, we choose the faces that shall admit a handle alternately and then we select their counterpart in the other sets of faces, corresponding to vertex w. This is illustrated in Figure 2.5. Each tube carries four edges, except the bottom handle, which carries three.

For $p \equiv 3 \bmod 4$ and $q \equiv 3 \bmod 4$, a similar construction is employed. Consider an embedding of $K_{p-1,q}$, where we note that $p-1 \equiv 2 \bmod 4$, thus even and q is odd, hence, $K_{p-1,q}$ has a quadrilateral embedding. We pick a vertex u in the partite set of size $p-1$ and split it in two vertices, v and w as in the previous case. The genus of $K_{p,q}$ is expected to be $\frac{(p-2)(q-2)+3}{4}$, thus, we can only use at most $\frac{q+1}{4}$ handles. Similarly to the previous case, we choose $\frac{q+1}{4}$ faces from each side in which handles are to be inserted. Each handle carries three edges, except for the bottom handle, which carries only two.

The same construction works for $p \equiv 3 \mod 4$ and $q \equiv 0 \mod 4$ or $p \equiv 1 \mod 4$ and $q \equiv 0 \mod 4$, however, only $\frac{q}{4}$ handles are required in both cases. Now, for $p \equiv 1 \mod 4, q \equiv 1 \mod 4$, we first obtain a quadrilateral embedding of $K_{p-3,q}$ since $p-3 \equiv 2 \mod 4$. Note that the genus of $K_{p-3,q}$ is $\frac{(p-5)(q-2)}{4}$. Three vertices are split and similarly to the above construction, we add $\frac{q-1}{4}$ handles near each split vertex, hence a total of $\frac{3(q-1)}{4}$ handles are added. In this case, the expected genus of $K_{p,q}$ is $\left\lceil \frac{(p-2)(q-2)+3}{4} \right\rceil = \left\lceil \frac{(p-5)(q-2)}{4} + \frac{3(q-1)}{4} \right\rceil$ which yields the desired embedding. All other combinations are covered by interchanging p and q. $\square$

3

Genus of several Cartesian Products

This chapter takes a closer look at the results concerning the Cartesian product of graphs, with particular emphasis on the repeated Cartesian product of bipartite graphs.

Restricted to the case when the Cartesian product is taken for two graphs, the results of this chapter can be summarised by the following theorem.

Theorem 3.0.1. $g(G \,\square\, H) = \begin{cases} 1 + 4r\ell(r+\ell-2), & \textit{if } G = K_{2r,2r} \textit{ and } H = K_{2\ell,2\ell}, \\ 0, & \textit{if } G = P_r \textit{ and } H = P_s, \textit{for } r \textit{ and } s \textit{ even}, \\ 1, & \textit{if } G = C_{2r} \textit{ and } H = C_{2s}. \end{cases}$

3.1 | General results

For a graph G with n vertices, m edges, and c components, the *first (graph) Betti number*, also called the cycle rank of G or the co-rank of G, denoted by $\mathscr{B}(G)$, is $m - n + c$. It is equal to the number of edges in the complement of a spanning tree for G, and thus gives the minimum number of edges that must be removed from G to remove all its cycles. The following results make use of this value.

We first give an upper bound for the genus of the Cartesian product of two graphs.

Theorem 3.1.1. *[29] The genus of $G_1 \square G_2$ satisfies the inequality*

$$g(G_1 \square G_2) \leq n_1 g(G_2) + n_2 g(G_1) + \mathscr{B}(K_{n_1,n_2}). \tag{3.1}$$

Proof. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ with $n_1 = |V_1|$ and $n_2 = |V_2|$. First note that the graph $G_1 \square G_2$ contains n_1 disjoint copies of G_2 and n_2 disjoint copies of G_1, where each vertex belongs to exactly one copy of G_1 and G_2.

Let $V(G_1 \square G_2) = \{(v_{i1}, v_{j2}) : i = 1, \ldots, n_1 \text{ and } j = 1, \ldots, n_2\}$. Let H_{j1} be the subgraph induced by $\{(v_{i1}, v_{j2}) : i = 1, \ldots, n_1\}$. Then, H_{j1} is isomorphic to G_1, for $j = 1, \ldots, n_2$. Similarly, let H_{i2} be the subgraph induced by $\{(v_{i1}, v_{j2}) : j = 1, \ldots, n_2\}$. Then, H_{i2} is isomorphic to G_2, for $i = 1, \ldots, n_1$. Note that in $G_1 \square G_2$, a vertex $(v_{i1}, v_{j2}) = H_{j1} \cap H_{i2}$. Refer to Figure 3.1 for an illustrated example.

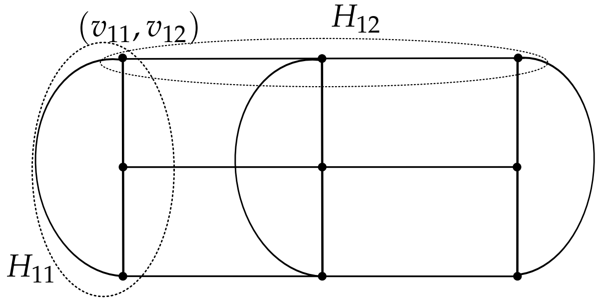


Figure 3.1: $K_3 \square P_3$

Another graph $G^* = (V^*, E^*)$ is formed by splitting each vertex (v_{i1}, v_{j2}) in $G_1 \square G_2$ into two vertices, v_{ij1} and v_{ij2}. Vertex v_{ij1} has the same adjacencies (v_{i1}, v_{j2}) had in H_{j1} and vertex v_{ij2} has the same adjacencies (v_{i1}, v_{j2}) had in H_{i2}. Since no new edges are added in G^*, $E^* = E(G_1 \square G_2)$. Therefore, G^* has n_1 copies of G_2 and n_2 copies of G_1 which are all mutually disjoint, as can be seen in Figure 3.2. Thus, by Theorem 1.2.10, $g(G^*) = n_1 g(G_2) + n_2 g(G_1)$. Let the components of G^* be separately embedded on orientable surfaces of appropriate genus.

To regain the original graph $G_1 \square G_2$, the surfaces are re-attached together in the following way. Consider vertex (v_{i1}, v_{j2}) which has been split into two new vertices, v_{ij1} and v_{ij2}, respectively contained in graphs H_{j1} and H_{i2}, which have been respectively embedded in orientable surfaces S_{j1} and S_{i2}. Take open 2-cells C_{j1} and C_{i2} in S_{j1} and S_{i2} respectively, with simple closed boundary curves J_{j1} and J_{i2} such that $(C_{j1} \cup J_{j1}) \cap H_{j1} = v_{ij1}$ and $(C_{i2} \cup J_{i2}) \cap H_{i2} = v_{ij2}$. Note that J_{j1} and J_{i2} do not contain the graphs H_{j1} and H_{i2}. Next, the surfaces S_{j1}, S_{i2} are amalgamated to each other forming a new surface S_{ij} by identifying J_{j1} and J_{i2}

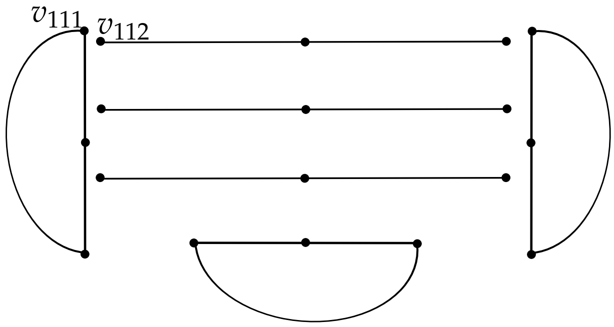


Figure 3.2: G^* consisting of disjoint copies of K_3 and P_3

such that v_{ij1} is identified with v_{ij2}. The new surface S_{ij} contains an embedding of the graphs H_{j1} and H_{i2} that share the vertex (v_{i1}, v_{j2}). The amalgamation process is illustrated in Figure 3.3.

There are $n_1 n_2$ identifications to be made to regain $G_1 \square G_2$. Consider the $n_1 + n_2$ surfaces on which the original components of G^* were embedded on to be the vertices in K_{n_1,n_2}. Therefore, each time two surfaces are identified to each other, this corresponds to adding an edge in K_{n_1,n_2}. Refer to Figure 3.4 for an illustration, where the dotted edge corresponds to amalgamating the two surfaces S_{11} and S_{12} from Figure 3.3. During this process, each time a cycle is formed in K_{n_1,n_2}, the genus is increased by at most one. This follows since some values of n_1 and n_2 produce no crossings for an embedding of K_{n_1,n_2} and so, there are no contributions to the genus. In other cases, a crossing is produced and hence, a new handle needs to be introduced, increasing the genus by one. Thus, the maximum number of additional handles required is exactly $\mathscr{B}(K_{n_1,n_2})$. Therefore, $g(G_1 \square G_2) \leq n_1 g(G_2) + n_2 g(G_1) + \mathscr{B}(K_{n_1,n_2})$, as required. □

The following theorem establishes a lower bound on the genus of the Cartesian product of two graphs.

Theorem 3.1.2. *[29] The genus of $G_1 \square G_2$ satisfies the inequality*

$$g(G_1 \square G_2) \geq \max(n_1 g(G_2) + g(G_1), n_2 g(G_1) + g(G_2)). \tag{3.2}$$

Proof. Using the same notation as in the previous result, let H be the graph formed by taking the union of H_{11} and H_{i2}, for $i = 1, \ldots, n_1$. Then, H is a subgraph of $G_1 \square G_2$ and the blocks of H are made up of one copy of G_1 and n_1

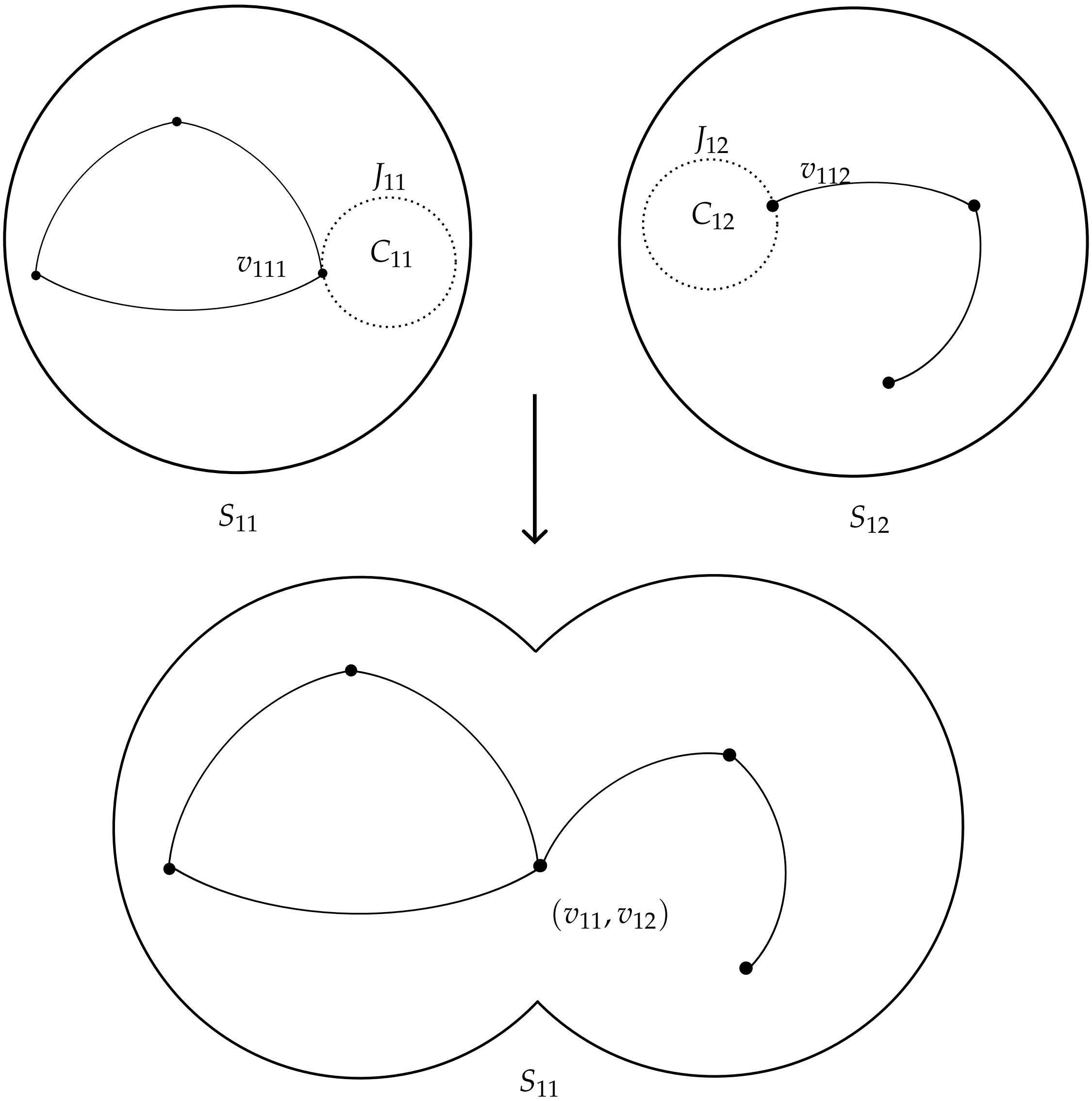


Figure 3.3: Amalgamating a copy of K_3 to a copy of P_3 from G^*

copies of G_2. Theorem 1.2.10 is applied to H to yield $g(H) = g(G_1) + n_1 g(G_2) \leq g(G_1 \square G_2)$. Similarly, since $G_1 \square G_2 = G_2 \square G_1$, we get $g(G_2) + n_2 g(G_1) \leq g(G_1 \square G_2)$. Hence, the result follows. $\square$

The two inequalities (3.1) and (3.2) coincide if and only if one of the graphs

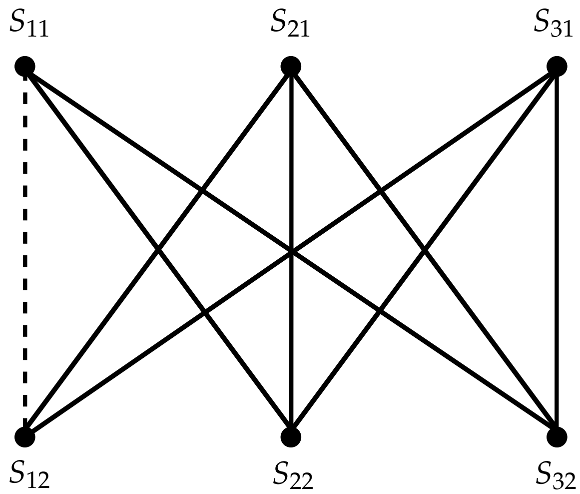


Figure 3.4: $K_{3,3}$ corresponding to surfaces induced by $K_3 \square P_3$

G_1 or G_2 is the trivial graph K_1, as discussed below. If

$$n_1 g(G_2) + n_2 g(G_1) + \mathscr{B}(K_{n_1,n_2}) = \max(n_1 g(G_2) + g(G_1), n_2 g(G_1) + g(G_2)) \tag{3.3}$$

then there are two cases to consider depending on the outcome of the right-hand side of (3.3). Suppose, $\max(n_1 g(G_2) + g(G_1), n_2 g(G_1) + g(G_2)) = n_1 g(G_2) + g(G_1)$, then by (3.3),

$$\begin{aligned} n_1 g(G_2) + g(G_1) &= n_1 g(G_2) + n_2 g(G_1) + n_1 n_2 - n_1 - n_2 + 1 \\ &\Leftrightarrow g(G_1) = n_2 g(G_1) + n_1 n_2 - n_1 - n_2 + 1 \\ &\Leftrightarrow g(G_1) = \frac{n_1 + n_2 - n_1 n_2 - 1}{n_2 - 1} \\ &\Leftrightarrow g(G_1) = \frac{n_1(1 - n_2) + n_2 - 1}{n_2 - 1} \\ &\Leftrightarrow g(G_1) = \frac{(n_2 - 1)(1 - n_1)}{n_2 - 1} \\ &\Leftrightarrow g(G_1) = 1 - n_1 \end{aligned}$$

Since the genus of a graph cannot be negative, the only possible value is $n_1 = 1$, therefore, $G_1 = K_1$. The case when $\max(n_1 g(G_2) + g(G_1), n_2 g(G_1) + g(G_2)) = n_2 g(G_1) + g(G_2)$ is analogous.

Before we proceed to determine the genus of some families of Cartesian products of bipartite graphs, we present some definitions on graph colouring.

Definition 3.1.3. *A (proper)* vertex-colouring *is a function $c : V(G) \to \mathbb{N}$ such that $c(u) \neq c(v)$ if u and v are adjacent in G.*

If each colour used is one of k given colours, then we refer to the colouring as a *k-colouring*. A graph G is *k-colourable* if there exists a colouring of G from a set of k colors. In other words, G is k-colourable if there exists a k-colouring of G.

Definition 3.1.4. *The minimum positive integer k for which G is k-colourable is the* chromatic number *of G and is denoted by $\chi(G)$.*

The chromatic number of bipartite graphs is known to be two. This is in fact another characterisation of bipartite graphs as stated in the following theorem.

Theorem 3.1.5. *[4] A non-empty graph G has chromatic number two if and only if G is a bipartite graph.*

The next result is a corollary of a result proved by Sabadussi in 1957 [22], where it was shown that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. We provide a new approach to proving directly the same result.

Proposition 3.1.6. *Let G and H be two non-empty graphs. The Cartesian product $G \square H$ is bipartite if and only if G and H are bipartite.*

Proof. Let G and H be two bipartite graphs with $V(G) = A \dot\cup B$ and $V(H) = C \dot\cup D$. Then, $V(G \square H) = (A \times C) \dot\cup (A \times D) \dot\cup (B \times C) \dot\cup (B \times D)$. To show that $G \square H$ is bipartite, we select an arbitrary vertex from each set in $V(G \square H)$, determine its adjacencies and show $G \square H$ has two partite sets.

- Consider the vertex $(a, c) \in A \times C$. Then (a, c) is adjacent to another vertex (x, y) in $V(G \square H)$ if and only if either $x = a$ and $y \sim c$ in H, or $y = c$ and $x \sim a$ in G. Therefore, $(x, y) \in A \times D$ or $(x, y) \in B \times C$.

- Consider the vertex $(a, d) \in A \times D$. Then (a, d) is adjacent to another vertex (x, y) in $V(G \square H)$ if and only if either $x = a$ and $y \sim d$ in H, or $y = d$ and $x \sim a$ in G. Therefore, $(x, y) \in A \times C$ or $(x, y) \in B \times D$.

- Consider the vertex $(b,c) \in B \times C$. Then (b,c) is adjacent to another vertex (x,y) in $V(G\Box H)$ if and only if either $x = b$ and $y \sim c$ in H, or $y = c$ and $x \sim b$ in G. Therefore, $(x,y) \in B \times D$ or $(x,y) \in A \times C$.

- Consider the vertex $(b,d) \in B \times D$. Then (b,d) is adjacent to another vertex (x,y) in $V(G\Box H)$ if and only if either $x = b$ and $y \sim d$ in H, or $y = d$ and $x \sim b$ in G. Therefore, $(x,y) \in B \times C$ or $(x,y) \in A \times D$.

From the above, we conclude that $G\Box H$ has a partite set consisting of vertices from $(A \times C)\dot{\cup}(B \times D)$ and another partite set consisting of vertices from $(A \times D)\dot{\cup}(B \times C)$. Therefore, $G\Box H$ is bipartite.

Conversely, let $G\Box H$ be a bipartite graph, where G has n_1 vertices and H has n_2 vertices. Then, $G\Box H$ is 2-chromatic by Theorem 3.1.5 and all subgraphs are 2-colourable, in particular, for $i = 1,\ldots,n_2$ and for $j = 1,\ldots,n_1$, G_i and H_j are 2-colourable. Suppose G_i is 1-colourable, then, $G_i = \bigcup_{k=1}^{n_1} K_1$, that is, G_i is empty, a contradiction. Similarly, if H_j is 1-colourable, we get a contradiction. Hence, G_i and H_j are 2-chromatic for $i = 1,\ldots,n_2$ and for $j = 1,\ldots,n_1$. Therefore, G and H are bipartite. $\square$

By induction, the result from Proposition 3.1.6 can be generalised to the repeated Cartesian product of k bipartite graphs.

Corollary 3.1.7. *For $i = 1,\ldots,k$, if G_i is bipartite, then the graphs H_k are bipartite, where $H_1 = G_1$ and $H_k = H_{k-1}\Box G_k$, for $k \geq 2$.*

3.2 | Genus of $K_{s,s}\Box K_{s,s}$

Using Proposition 2.1.3 together with Theorem 2.0.1, Remark 2.1.4 and Lemma 2.1.5, we are in a position to derive the genus of $K_{2r,2r}\Box K_{2r,2r}$.

Lemma 3.2.1. *[28] The genus of $K_{2r,2r}\Box K_{2r,2r}$ is given by*

$$g(K_{2r,2r}\Box K_{2r,2r}) = 1 + 8r^2(r-1).$$

Proof. By Proposition 3.1.6, $K_{2r,2r}\Box K_{2r,2r}$ is a bipartite graph. We proceed by constructing a quadrilateral embedding for $K_{2r,2r}\Box K_{2r,2r}$, for $r \geq 2$. First embed $4r$

disjoint copies of $K_{2r,2r}$ in $4r$ surfaces of genus $(r-1)^2$ using the embedding given in Remark 2.1.4. For half of these copies, that is, for $2r$ of them, assign one orientation and give the reverse orientation to the other half. This separation corresponds to the partite sets of $K_{2r,2r}$. All that is left is to add edges between the $4r$ disjoint copies to acquire $K_{2r,2r}\square K_{2r,2r}$. Note that $4r$ edges are required to be added between each pair of oppositely oriented copies, which can be done by adding r handles, with each handle carrying four edges. Such an operation will be referred to as a link. For each surface, there are $2r$ links. By Lemma 2.1.5, the embedding for each copy of $K_{2r,2r}$ on each surface can be applied. We just need to check that after partitioning the faces, the corresponding subsets can be matched correctly. For $1 \leq j \leq 2r$, each copy j of $K_{2r,2r}$ is embedded with common orientation, whilst for $1 \leq \overrightarrow{j} \leq 2r$ each copy $\overrightarrow{j}$ of $K_{2r,2r}$ is embedded with the reverse orientation. For some copy j, its face partition consists of $2r$ subsets of r faces each. Match subset i of the face partition to subset $\overrightarrow{i}$ in copy $\overrightarrow{j}+\overrightarrow{i}, 1 \leq \overrightarrow{j}+\overrightarrow{i} \leq 2r \pmod{2r}$. If $\overrightarrow{j}+\overrightarrow{i} = \overrightarrow{j}'+\overrightarrow{i}'$ with $\overrightarrow{i} = \overrightarrow{i}'$, then, $j = j'$, therefore, each subset of the partition has exactly one handle attached at each face in that subset. Each handle carries maximum four edges, so that each new face formed is quadrilateral. Since a quadrilateral embedding has been constructed, Proposition 2.1.3 can be applied. As $n = 4r.4r = 16r^2$ and $m = 4r.4r^2 + 4r.4r^2 = 32r^3$, the genus $g(K_{2r,2r}\square K_{2r,2r}) = 1 + 8r^2(r-1)$. The case when $r = 1$, the construction used above cannot be employed by Remark 2.1.6. Note that $K_{2,2}\square K_{2,2}$ is $C_4\square C_4$, which is known to be toroidal [28], which conforms to the above result. □

Two similar results hold for $K_{1,1}\square K_{1,1}$ and for $K_{3,3}\square K_{3,3}$.

Lemma 3.2.2. *[28] The genus of $K_{3,3}\square K_{3,3}$ is*

$$g(K_{3,3}\square K_{3,3}) = 10.$$

Proof. An embedding of $K_{3,3}$ can have faces with six sides each such that

$$N_{K_{3,3}}(i) = \begin{cases} \{4,5,6\}, & \text{for } 1 \leq i \leq 3, \\ \{1,2,3\}, & \text{for } 4 \leq i \leq 6 \end{cases}$$

and permutations given by

$$P_1, P_2, P_3 : (4\ 6\ 5)$$
$$P_4, P_5, P_6 : (1\ 3\ 2).$$

This embedding produces six-sided faces that contain each vertex of the graph exactly once. Now, six copies of $K_{3,3}$ are embedded in six surfaces of genus one, where again half are given one orientation and the other half the reverse orientation. Nine handles are added, each carrying six edges as described for the previous case. Therefore, a quadrilateral embedding is obtained and computing the genus using Proposition 2.1.3, we get $g(K_{3,3}\square K_{3,3}) = 10$. □

Lemma 3.2.3. *[28] The genus of $K_{1,1}\square K_{1,1}$ is*

$$g(K_{1,1}\square K_{1,1}) = 0.$$

Proof. $K_{1,1}\square K_{1,1} = K_2\square K_2 = C_4$ and thus, $g(C_4) = 0$. □

Lemmas 3.2.1, 3.2.2 and 3.2.3 are summarised in the following theorem.

Theorem 3.2.4. *[28] The genus of $K_{s,s}\square K_{s,s}$ if s is even, or if $s \in \{1, 3\}$ is given by*

$$g(K_{s,s}\square K_{s,s}) = 1 + s^2(s-2).$$

Theorem 3.2.4 can be used to produce the following corollary, which is a slight generalisation of the theorem.

Corollary 3.2.5. *[28] The genus of $K_{2r,2r}\square K_{t,s}$ is given by*

$$g(K_{2r,2r}\square K_{t,s}) = 1 + r((r-2)(t+s) + ts)$$

for $t \leq 2r, s \leq 2r$.

Proof. First, $K_{2r,2r}\square K_{2r,2r}$ is embedded as shown in Theorem 3.2.4. Next, $4r - (t+s)$ copies of $K_{2r,2r}$ are removed, including any handles included in these copies, to leave an embedding of $K_{2r,2r}\square K_{t,s}$. This embedding is also quadrilateral since the embedding for $K_{2r,2r}\square K_{2r,2r}$ was quadrilateral and the removal of handles simply reintroduces quadrilateral faces. Now, the number of vertices n, for $K_{2r,2r}\square K_{t,s}$, is $4r(t+s)$ whilst the number of edges m is $4r^2(t+s) + 4rts$. Applying Proposition 2.1.3, the result immediately follows. □

To further generalise Theorem 3.2.4, we use Corollary 3.2.5 to obtain an unbounded formula.

Corollary 3.2.6. *[28] The genus of $K_{2r,2r}\Box K_{2\ell,2\ell}$ is given by*

$$g(K_{2r,2r}\Box K_{2\ell,2\ell}) = 1 + 4r\ell(r + \ell - 2)$$

for all natural numbers r, ℓ.

Proof. In Corollary 3.2.5, let $s = 2\ell = t$, therefore,

$$\begin{aligned} g(K_{2r,2r}\Box K_{2\ell,2\ell}) &= 1 + r((r-2)(2\ell + 2\ell) + 4\ell^2) \\ &= 1 + 4r(r\ell - 4\ell + \ell^2) \\ &= 1 + 4r^2\ell - 8r\ell + 4r\ell^2 \\ &= 1 + 4r\ell(r + \ell - 2) \end{aligned}$$

The result holds for all natural numbers r and ℓ. $\Box$

Work on the genus of the repeated Cartesian product of bipartite graphs also includes the work of Pisanski [18] who generalised White's results by using the degree sequence of regular bipartite graphs to determine the genus of their repeated Cartesian product. The main result is given in the following theorem, the proof of which is ommitted here.

Theorem 3.2.7. *[18] Let $G(n,d)$ denote a connected regular bipartite graph on $2n$ vertices and of degree d. The main result states that any Cartesian product $G(n,d)\Box G_1(n_1,d_1)$ $\Box G_2(n_2,d_2)\Box \ldots \Box G_m(n_m,d_m)$, such that $\max\{d_1,d_2,\ldots,d_m\} \leq d \leq d_1 + d_2 + \ldots + d_m$, has a quadrilateral embedding.*

Whilst this result captures several families of regular bipartite graphs, it cannot be applied to the family of graphs $K_{2r,2r}\Box C_{2s}$. Extending the construction provided by White, we derive and prove a formula for the genus of $K_{2r,2r}\Box C_{2s}$.

Theorem 3.2.8. *For $r \geq 2$, the graph $G = K_{2r,2r}\Box C_{2s}$ has a quadrilateral embedding on a surface of genus g.*

Proof. Consider a surface S of genus g. We start by embedding $2s$ copies of $K_{2r,2r}$, each on a surface of genus $(r-1)^2$ as described by Theorem 3.2.4. We shall construct a quadrilateral embedding of G on S for an appropriate value of g. We partition the $2s$ copies in two such that s copies, labelled by $1 \leq j \leq s$, are assigned one orientation and the remaining s copies, labelled $1 \leq \overrightarrow{j} \leq s$, are assigned the reverse orientation, corresponding to the vertex-set partition of C_{2s}. From Lemma 2.1.5, each copy of $K_{2r,2r}$ has a partitioning of its faces into $2r$ subsets of r faces each. For this case, only $2r$ faces are needed, therefore, we select any two subsets for some copy of $K_{2r,2r}$. This selection of faces is then applied to all the copies of $K_{2r,2r}$. Two links need to be made from each copy j, both links to a copy of reverse orientation. Each link consists of $4r$ edges, which can be passed over r handles, each handle carrying four edges. The handles corresponding to the first link are inserted in the faces of one subset of copy j of $K_{2r,2r}$ to the corresponding faces in copy $\overrightarrow{j}$, whilst the handles for the second link are attached to the faces of the second subset of copy j to the corresponding faces in copy $\overrightarrow{j}$. In this way, after all the $2s$ links have been made, we have obtained a quadrilateral embedding of $K_{2r,2r}\Box C_{2s}$. $\square$

Remark 3.2.9. *Note that for $K_{2,2}\Box C_{2s}$, this graph is in fact $C_4\Box C_{2s}$. The quadrilateral embedding of this graph is called the grid graph or lattice graph and is known to be toroidal [28]. Therefore, $g(K_{2,2}\Box C_{2s}) = 1$.*

Using this result, we are now in a position to derive the genus of $K_{2r,2r}\Box C_{2s}$.

Theorem 3.2.10. *For $r \geq 1$, the genus of $K_{2r,2r}\Box C_{2s}$ is given by*

$$g(K_{2r,2r}\Box C_{2s}) = 1 + 2rs(r-1).$$

Proof. By Corollary 3.1.7, $K_{2r,2r}\Box C_{2s}$ is a bipartite graph, If $r \geq 2$, by Theorem 3.2.8, there is a quadrilateral embedding of the same graph. Hence, we apply Proposition 2.1.3 to get

$$\begin{aligned} g(K_{2r,2r}\Box C_{2s}) &= 1 + \frac{4r(2s) + 2s(4r^2)}{4} - \frac{8rs}{2} \\ &= 1 + 2rs + 2r^2s - 4rs \\ &= 1 + 2rs(r-1). \end{aligned}$$

In the case when $r = 1$, the genus of $K_{2,2}\Box C_{2s}$ is one by Remark 3.2.9, which conforms to the above formula. □

As a direct consequence of Theorem 3.2.10, we deduce the genus of $K_{2r,2r}\Box P_{2s}$.

Corollary 3.2.11. *For $r \geq 1$, the genus of $K_{2r,2r}\Box P_{2s}$ is given by*

$$g(K_{2r,2r}\Box P_{2s}) = (r-1)(2sr-1).$$

Proof. We start by first embedding $G' = K_{2r,2r}\Box C_{2s}$ as given in Theorem 3.2.8. To obtain $G = K_{2r,2r}\Box P_{2s}$, we only need to remove $4r$ edges from G' between a copy of $K_{2r,2r}$ with one orientation and another copy of reverse orientation. The only way these two copies are connected is through r handles, therefore, we remove these r handles and the edges on them. The faces we have reintroduced are quadrilateral, since the embedding of G' is quadrilateral. Therefore, we have obtained a quadrilateral embedding for G. By Corollary 3.1.7, G is bipartite, hence, we apply Proposition 2.1.3 to get:

$$\begin{aligned} g(K_{2r,2r}\Box P_{2s}) &= 1 + 2rs(r-1) - r \\ &= 2rs(r-1) - (r-1) \\ &= (r-1)(2rs-1). \end{aligned}$$

□

3.3 | Genus of H_j

Every path is a bipartite graph, thus, the Cartesian product of two paths is again bipartite by Proposition 3.1.6. This can be extended to the Cartesian product of j paths, which for $j \geq 2$, is defined recursively as follows

$$H_1 = P_{m_1}$$

$$H_j = H_{j-1}\Box P_{m_j}.$$

Whilst all paths are bipartite, we restrict the first three paths to be on an even number of vertices. The reason for this is to obtain a quadrilateral embedding in the base case, that is, when $j = 3$, as will become evident in the proof. Let $M^{(j)} = \prod_{i=1}^{j} m_i$ and $m^{(i)} = \sum_{i=1}^{j} \frac{1}{m_i}$. The following theorem establishes the genus of H_j, where the first three paths in the product are all on an even number of vertices.

Theorem 3.3.1. *[28] The genus of H_j, for $j \geq 3$ and m_1, m_2 and m_3 all even, is given by*

$$g(H_j) = 1 + \frac{M^{(j)}(j - 2 - m^{(j)})}{4}.$$

Proof. By Corollary 3.1.7, H_j is a bipartite graph, hence, we proceed to construct a quadrilateral embedding for it in order to compute its genus by using Proposition 2.1.3. Note that the number of vertices of H_j is $M^{(j)}$. Let $P(j)$ be the property that there is an embedding of H_j, for which each face is quadrilateral, with $f_j = \frac{M^{(j)}(j-m^{(j)})}{2}$, including two disjoint sets of $\frac{M^{(j)}}{4}$ mutually vertex-disjoint quadrilateral faces each, both sets containing all $M^{(j)}$ vertices of H_j, with the number of edges being $M^{(j)}(j - m^{(j)})$. The proof of $P(j)$ will proceed by induction on j, for $j \geq 3$.

For a construction of H_3, first consider the graph $P_6 \square P_8$, which is known to be a planar graph, as can be seen in Figure 3.5. Every face is quadrilateral, except for the exterior face. In general, the graph $P_{m_1} \square P_{m_2}$ is planar and two links may be made, provided both m_1 and m_2 are even. One link uses the faces designated by (1) and the other uses the faces designated by (2), as illustrated in Figure 3.5.

With m_3 being even, embed m_3 copies of $P_{m_1} \square P_{m_2}$ and partition them in two, with $\frac{m_3}{2}$ copies having on orientation, and the other half having reverse orientation. This partition corresponds to the vertex-set partition of P_{m_3}. Arrange the two end copies of $P_{m_1} \square P_{m_2}$ so that the faces designated by (2), including the exterior face, are used in the single link made from each end copy. These two end copies should be found in different parts of the partition. The graph H_3 thus has a quadrilateral embedding since a handle attached between two oppositely oriented copies of the exterior face replaces those two faces of $2(m_1 + m_2 - 2)$ sides each with $2(m_1 + m_2 - 2)$ quadrilaterals once the edges are added over the handle. Now, the number of edges in $P_{m_1} \square P_{m_2}$ is $(m_1 - 1)m_2 + (m_2 - 1)m_1$, which is equal to $2m_1m_2 - m_1 - m_2$. Therefore, the num-

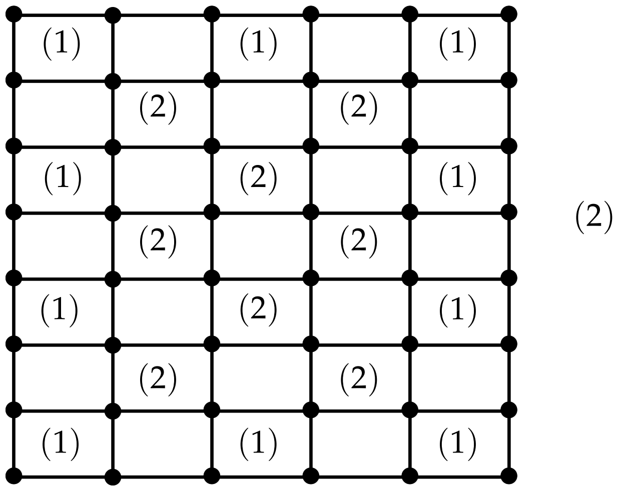


Figure 3.5: $P_6 \square P_8$ [29]

ber of edges in H_3 is $m_3(2m_1m_2 - m_1 - m_2) + (m_3 - 1)(m_1m_2)$, which is equal to $3m_1m_2m_3 - m_1m_3 - m_2m_3 - m_1m_2$. Furthermore, the number of faces f_3 in H_3 is given by $f_3 = m_3f_2 + \tilde{f}^{(1)} + \tilde{f}^{(2)}$, where $\tilde{f}^{(1)}$ denotes the increase in the number of faces produced by the added handles within faces designated by (1), similarly for $\tilde{f}^{(2)}$. From each copy of $P_{m_1} \square P_{m_2}$, there are $\frac{m_1}{2}\frac{m_2}{2}$ handles within faces of type (1), with $\frac{m_3}{2} - 1$ links that make use of these faces. Therefore, $\tilde{f}^{(1)} = 2(\frac{m_3}{2} - 1)(\frac{m_1}{2}\frac{m_2}{2})$. The faces designated by (2), excluding the exterior face, produce a change in faces of $2(\frac{m_1}{2} - 1)(\frac{m_2}{2} - 1)$, whilst the exterior face accounts for $2(m_1 + m_2 - 2) - 2$. Therefore, we have that

$$\begin{aligned} f_3 &= m_3((m_1 - 1)(m_2 - 1) + 1) + 2\left(\frac{m_3}{2} - 1\right)\left(\frac{m_1}{2}\frac{m_2}{2}\right) \\ &+ \frac{m_3}{2}\left(2\left(\frac{m_1}{2} - 1\right)\left(\frac{m_2}{2} - 1\right) + 2(m_1 + m_2 - 3)\right) \\ &= \frac{3}{2}m_1m_2m_3 - \frac{m_1m_2}{2} - \frac{m_1m_3}{2} - \frac{m_2m_3}{2}. \end{aligned}$$

To show that $P(3)$ holds, all that remains is to select two disjoint sets of quadrilateral faces that cover all the vertices of H_3. From each handle within faces marked by (2), select every second face. These are mutually vertex-disjoint and contain all $m_1m_2m_3$ vertices of H_3. The second set of faces is formed by choosing the remaining faces on the handles joining the faces designated by (2). These are also mutually vertex-disjoint and cover all $m_1m_2m_3$ vertices of H_3. Clearly, the two sets chosen are disjoint, hence $P(3)$ is true.

Assume $P(j)$ holds, we show that $P(j+1)$ follows. First, minimally embed m_{j+1} copies of H_j as described by $P(j)$, with orientation as determined by the vertex set partition of $P_{m_{j+1}}$. The $m_{j+1}-1$ required links can be made to obtain a quadrilateral embedding of H_{j+1} as described above. Now, the number of edges in H_{j+1} is m_{j+1} multiplied by the number of edges in H_n, added with the product of $m_{j+1}-1$ with the number of vertices in H_n. Thus,

$$m = m_{j+1}(jM^{(j)} - M^{(j)}m^{(j)}) + (m_{j+1}-1)M^{(j)} = M^{(j+1)}(j+1-m^{(j+1)}).$$

Also, $f_{j+1} = m_{j+1}f_j + \tilde{f}$, where $\tilde{f} = 2(m_{j+1}-1)(\frac{M^{(j)}}{4})$, where $m_{j+1}-1$ is the number of links, $\frac{M^{(j)}}{4}$ is the number of handles per link, with an increase of two in the number of faces for each handle. Therefore, we have,

$$\begin{aligned} f_{j+1} &= m_{j+1}\left(\frac{M^{(j)}(j-m^{(j)})}{2}\right) + \frac{M^{(j+1)}}{2} - \frac{M^{(j)}}{2} \\ &= \frac{M^{(j+1)}(j+1-m^{(j+1)})}{2}. \end{aligned}$$

The last step to verify $P(j+1)$ is to find two disjoint sets of $\frac{M^{(j+1)}}{4}$ mutually vertex-disjoint quadrilateral faces each, both sets containing all $M^{(j+1)}$ vertices of H_{j+1}. There are two cases to consider:

Case (i): If m_{j+1} is even, then we choose the opposite faces on each handle of alternate links to from one set, and the remaining faces form the second set. This selection is depicted in Figure 3.6.

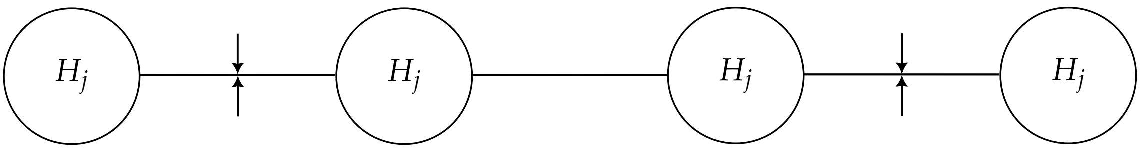


Figure 3.6: Selecting faces from an even number of paths [28]

Case (ii): If m_{j+1} is odd, from alternating links, choose the opposite faces on each handle together with the remaining set of $\frac{M^{(j)}}{4}$ mutually vertex-disjoint quadrilateral faces from one of the end copies of H_j. To form the other set, choose opposite faces from each handle on the rest of the links, together with the $\frac{M^{(j)}}{4}$

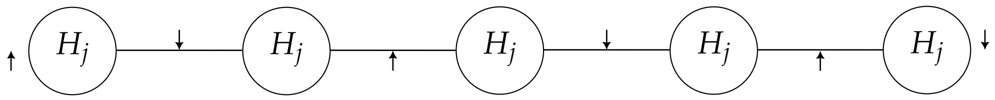


Figure 3.7: Selecting faces from an odd number of paths [28]

mutually vertex-disjoint quadrilateral faces from the remaining end copy of H_j. This selection is illustrated in Figure 3.7.

We have shown that $P(j+1)$ follows from $P(j)$, for all $j \geq 3$. Using Proposition 2.1.3, we can compute the genus of H_j as follows,

$$\begin{aligned} g(H_j) &= 1 + \frac{M^{(j)}(j - m^{(j)})}{4} - \frac{M^{(j)}}{2} \\ &= 1 + \frac{M^{(j)}(j - 2 - m^{(j)})}{4}. \end{aligned}$$

□

For the case when $m_i = k$, for all $i = 1, \ldots, j$, we get the following result.

Corollary 3.3.2. *[28] The genus of $H_j^{(k)}$, for $j \geq 3$ is given by*

$$g(H_j^{(k)}) = 1 + \frac{k^{j-1}(kj - 2k - j)}{4}.$$

3.4 | Genus of G_j

Every cycle on an even number of vertices is a bipartite graph, thus, the Cartesian product of two even cycles is again bipartite by Proposition 3.1.6. This can be extended to the Cartesian product of j even cycles. We denote the product of j even cycles, for $j \geq 2$ recursively as follows

$$G_1 = C_{2m_1}$$

$$G_j = G_{j-1} \square C_{2m_j}.$$

From the above definition, it is required that $m_i \geq 2$, for $i \in \{1, \ldots, j\}$ as K_2 is not considered to be a cycle. As in other sections in this chapter, let $M^{(j)} = \prod_{i=1}^{j} m_i$. The following lemma establishes the number of vertices and edges in G_j.

Lemma 3.4.1. *The number of vertices of G_j is $2^j M^{(j)}$ and the number of edges is $j2^j M^{(j)}$.*

Proof. The number of vertices of G_j is the product of the vertices of each even cycle in G_j. Since all cycles are a multiple of two, then the number of vertices of G_j will contain 2^j. Moreover, $m_1, m_2, \ldots, m_j$ are also multiplied together to finally obtain $n = 2^j M^{(j)}$.

To deduce the number of edges of G_j, the proof will proceed by induction on j.

For $j = 2$, $G_2 = C_{2m_1} \Box C_{2m_2}$. Then, $m = 2m_1 2m_2 + 2m_1 2m_2 = 2.2^2 m_1 m_2$. Therefore, the result holds for the base case.

Suppose the result holds for $j = k-1$, we will show it is true for $j = k$. The number of edges of G_k is

$$\begin{aligned} m &= |V(G_{k-1})|.|E(C_{2m_k})| + |V(C_{2m_k})|.|E(G_{k-1})| \\ &= 2^{k-1}M^{(k-1)}.2m_k + 2m_k.(k-1)2^{k-1}M^{(k-1)} \text{by the induction hypothesis} \\ &= 2^k M^{(k)} + (k-1)2^k M^{(k)} \\ &= k2^k M^{(k)} \end{aligned}$$

The result follows by induction on j. □

Now we can proceed to prove the genus of the repeated Cartesian product of even cycles.

Theorem 3.4.2. *[28] The genus of G_j, for $j \geq 2$, is given by*

$$g(G_j) = 1 + 2^{j-2}(j-2)M^{(j)}.$$

Proof. By Corollary 3.1.7, G_j is a bipartite graph, hence, we proceed to construct a quadrilateral embedding for it to compute its genus using Proposition 2.1.3. Let $P(j)$ be the property that there is an embedding of G_j, for which each face is quadrilateral, with $f_j = j2^{j-1}M^{(j)}$, including two disjoint sets of $2^{j-2}M^{(j)}$ mutually vertex-disjoint quadrilateral faces each, both sets containing all $2^j M^{(j)}$ vertices of G_j. The proof of $P(j)$ will proceed by induction on j, for $j \geq 2$.

For $G_2 = C_{2m_1} \Box C_{2m_2}$, take $m_1 = 2$ and $m_2 = 3$ to yield $G_2 = C_4 \Box C_6$. This graph is embedded with no crossings on the torus, that is, the genus is one, thus, it conforms to the expected value. The embedding is known as the grid graph or lattice graph, which can be seen in Figure 3.8.

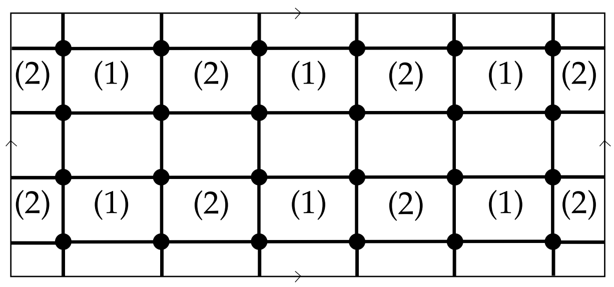


Figure 3.8: $C_4 \square C_6$

The number of quadrilateral faces $f_j = 2.2.M^{(2)} = 4.2.3 = 24$ can also be verified from the embedding, noting also that faces marked by (1) and (2) form the two sets of faces as described by $P(2)$.

Assuming $P(j)$ is true, we will show that $P(j+1)$ also holds. First, minimally embed $2m_{j+1}$ copies of G_j as described by $P(j)$. Assign m_{j+1} many copies one orientation and the other m_{j+1} copies the reverse orientation. This partition corresponds to the vertex set partition of $C_{2m_{j+1}}$. From each copy of G_j, two links are made to copies of opposite orientation. From $P(j)$, we can construct these two links using $2^{j-2}M^{(j)}$ handles, where each handle carries four edges. Therefore, each new face formed is a quadrilateral face. In this way, G_{j+1} can be embedded after constructing the $2m_{j+1}$ links.

Now, form one set of faces by choosing alternating quadrilaterals from each handle, which was added in alternate links in this embedding. Form the second set by choosing the rest of the quadrilaterals on the same handles. The two sets of faces are disjoint and each set contains $2(m_{j+1})(2^{j-2}M^{(j)}) = 2^{j-1}M^{(j+1)}$ mutually vertex-disjoint quadrilaterals. This follows because links are chosen alternately, therefore, from the $2m_{j+1}2^{j-2}M^{(j)}$ we choose half of them. From this number of handles, which is $m_{j+1}2^{j-2}M^{(j)}$, a set is formed by again choosing faces alternately, thus, we select only two faces from each handle. Note that both sets contain all of the $2^{j+1}M^{(j+1)}$ vertices of G_{j+1}. Furthermore, the number of faces in G_{j+1} is the number of faces in each copy of G_j and the net increase in the number of faces introduced by the handles. Therefore, $f_{j+1} = 2m_{j+1}f_j + \tilde{f}$, where $\tilde{f}$ is the contribution to the number of faces from the handles. By the induction hypothesis, $f_j = j2^{j-1}M^{(j)}$. To calculate $\tilde{f}$, we know that there are $(2m_{j+1})(2^{j-2}M^{(j)})$ handles in total in G_{j+1}, however, even though each handle introduces four new faces, the net difference in the total number of faces is two per handle. This follows since first we have two faces in two disjoint copies of G_j, then, a handle connects them together and we get four faces, so there is only an increase of two in the number of faces. Therefore, $\tilde{f} = (2m_{j+1})(2^{j-2}M^{(j)})(2) = 2^jM^{(j+1)}$. Hence, $f_{j+1} = (j+1)2^jM^{(j+1)}$. We have shown that $P(j+1)$ follows from $P(j)$. Using Proposition 2.1.3, we can compute the genus of G_j.

$$\begin{aligned} g(G_j) &= 1 + \frac{j2^jM^{(j)}}{4} - \frac{2^jM^{(j)}}{2} \\ &= 1 + 2^{j-2}(j-2)M^{(j)}. \end{aligned}$$

□

If $m_i = k$ for all $i \in \{1, \ldots, j\}$, then, let $G_j^{(k)}$ denote the repeated Cartesian product of cycles, all of which have length k. Substituting $M^{(j)} = k^j$ in Theorem 3.4.2 we obtain the following result.

Corollary 3.4.3. *[28] The genus of $G_j^{(k)}$ is given by $g(G_j^{(k)}) = 1 + 2^{j-2}(j-2)k^j$.*

3.5 | Genus of the j-cube and other variants

Definition 3.5.1. *The j-*cube*, denoted by Q_j, is a graph on 2^j vertices, such that each vertex v is a sequence $a_1a_2\ldots a_j$ of length j where a_i is either 0 or 1. An edge $\{u, v\}$ is in Q_j if the sequences corresponding to u and v differ only at one position.*

The genus of the j-cube was found by Ringel in 1955 [19] and independently by Beineke and Harary in 1965 [1]. Since the j-cube can be represented as the repeated Cartesian product of K_2, the result can also be obtained as a corollary of Theorem 3.5.5. Similarly, it is a corollary of Corollary 3.3.2 since $K_2 = P_2$ and thus, the j-cube is represented as the repeated Cartesian product of P_2. The following gives the formula for its genus.

Theorem 3.5.2. *[1, 19, 28] The genus of the j-cube is given by*

$$g(Q_j) = 1 + 2^{j-3}(j-4).$$

The following is the proof given by Beineke and Harary in [1], which makes use of quadrilateral embeddings.

Proof. We derive the formula for the genus of Q_j by showing the following two inequalities:

$$g(Q_j) \geq 1 + 2^{j-3}(j-4) \tag{3.4}$$

$$g(Q_j) \leq 1 + 2^{j-3}(j-4) \tag{3.5}$$

To show (3.4), we use Corollary 1.2.8 and the fact that $4f \geq 2m$ since Q_j has no odd cycles, particularly of length three. Therefore, $g(G) \geq 1 + \frac{m}{4} - \frac{n}{2}$. Applying this to Q_j, with $n = 2^j$ and $m = j2^{j-1}$ we get $g(Q_j) \geq 1 + 2^{j-3}(j-4)$, satisfying (3.4).

To show (3.5) holds, we prove the following claim.

Claim: Q_j can be embedded on an orientable surface of genus g in such a way that every quadrilateral in Q_j whose vertices differ only in the first and $(j-1)$th places of their sequences is a face.

Proof: The proof proceeds by induction on j. The result clearly holds for Q_2.

Assume it is true for Q_{j-1}. Embed two copies of Q_{j-1} on two orientable surfaces S and S' of appropriate genus, given by the claim. These are embedded in such a way that the two embeddings are "mirror images" of each other and they satisfy the conditions of the claim. In both S and S', each vertex v lies on exactly one quadrilateral face $F(v)$ whose vertices differ only in their first and $(j-1)$th place.

We now construct a quadrilateral embedding for Q_j. Suffix each sequence of each vertex in S by a 0 and similarly, suffix each sequence of each vertex in S' by a 1. Place a handle from $F(00\ldots00)$ in S to $F(00\ldots01)$ in S' to obtain a surface of genus $2g(Q_{j-1})$ and repeat this for all such faces. There are $\frac{(2^{j-1}-4)}{4}$ other faces on S which are faces $F(v)$ for four vertices v. This is the case as for Q_{j-1}, there are 2^{j-1} vertices, each of which lies in a face $F(v)$. We subtract four since we have already considered one such face and we divide by four because each face is made up of four vertices and otherwise, we would be counting a face four times. Place handles between corresponding faces of S and S'. Therefore, we add $2^{j-3} - 1$ handles in total. The surface S obtained from S and S' has genus $2g(Q_{j-1}) + 2^{j-3} - 1$. We note that S can have Q_j embedded on it as vertices from S are to be joined to vertices from S' without intersections using the handles as they only differ at the jth position. Furthermore, each quadrilateral of Q_j is a face. ■

Note that S has genus $(j-5)2^{j-2} + 2 + 2^{j-3} - 1 = (1 + j - 5)2^{j-3} + 1 = (j-4)2^{j-3} + 1$ as required to show. □

Below is an illustration of the technique described in the proof for Theorem 3.5.2 given by Beineke and Harary [1].

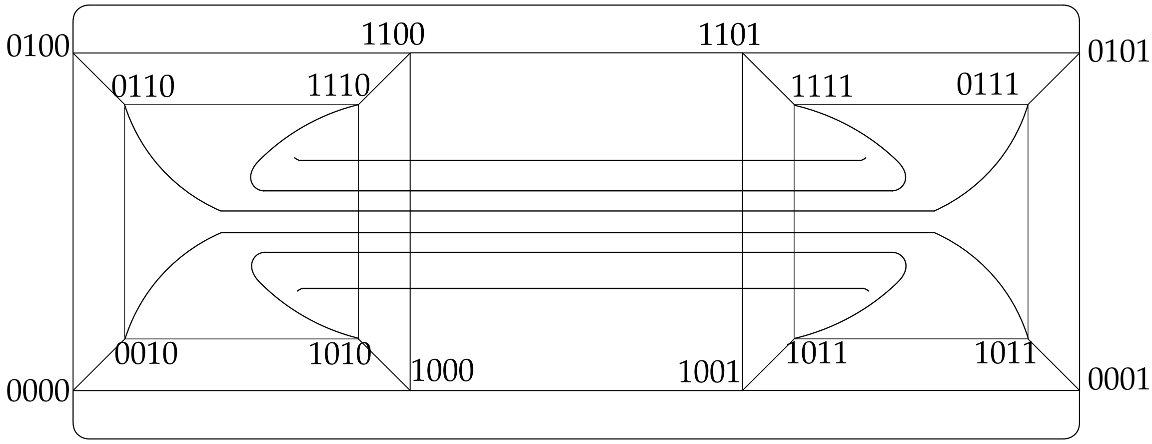


Figure 3.9: A quadrilateral embedding of Q_4 from two copies of Q_3 [1]

We now define a generalisation of the j-cube by letting $Q_1^{(s)} = K_{s,s}$ and recursively define $Q_j^{(s)} = Q_{j-1}^{(s)} \square K_{s,s}$, for $j \geq 2$. The constructions developed in Theorem 3.2.4 are extended to obtain the genus of $Q_j^{(2s)}$. First, we derive the number of vertices and edges in $Q_j^{(2s)}$.

Lemma 3.5.3. *For $Q_j^{(2s)}$, the number of vertices is $4^j s^j$ and the number of edges is $j2^{2j}s^{j+1}$.*

Proof. The number of vertices in $Q_j^{(2s)}$ is the product of the vertices of each $K_{2s,2s}$. Therefore, $n = (4s)^j = 4^j.s^j$.

The degree of a vertex v in $Q_j^{(2s)}$ is $2sj$ as v is in j copies of $K_{2s,2s}$, and for each copy it is adjacent to $2s$ vertices. Therefore, by the Handshaking Lemma Lemma 1.1.1, we get

$$\begin{aligned} 2m = \sum_{v \in Q_j^{(2s)}} deg(v) &= (2sj)(4^j s^j) \\ &= j2^{2j+1}s^{j+1} \\ \Rightarrow m = j2^{2j}s^{j+1}. \end{aligned}$$

□

Theorem 3.5.4. *[28] The genus of $Q_j^{(2s)}$ is given by*

$$g(Q_j^{(2s)}) = 1 + 2^{2j-2}s^j(sj - 2).$$

Proof. By Corollary 3.1.7, $Q_j^{(2s)}$ is a bipartite graph. We proceed to construct a quadrilateral embedding and then apply Proposition 2.1.3 to compute the genus. By Lemma 3.5.3, $n = 4^j s^j$ and $m = j2^{2j}s^{j+1}$. The result will be proved by induction on the statement $P(j)$ which states that there is a quadrilateral embedding of $Q_j^{(2s)}$ such that the number of faces $f_j = j2^{2j-1}s^{j+1}$, including $2r$ disjoint sets of $2^{2j-2}s^j$ mutually vertex-disjoint quadrilateral faces each, where each set contains all $4^j s^j$ vertices of $Q_j^{(2s)}$.

The case for $Q_1^{(2s)}$ is the embedding of $K_{2s,2s}$ given by Theorem 2.0.1, Remark 2.1.4 and Lemma 2.1.5.

Assuming $P(j)$ to be true, we will show that $P(j+1)$ holds. First embed $4s$ copies of $Q_j^{(2s)}$ as described by $P(j)$. Assign $2s$ of these copies one orientation, with labelling j, where $1 \leq j \leq 2s$ and the other $2s$ the reverse orientation, labelled $\overrightarrow{j}$, where $1 \leq \overrightarrow{j} \leq 2s \pmod{2s}$. This partition corresponds to the vertex-set partition of $K_{2s,2s}$. By the induction hypothesis, each copy of $Q_j^{(2s)}$ has $2s$ sets of faces. Each set is used for a link, thus, we have $2s$ links made from each copy. Furthermore, each set contains all the vertices of $Q_j^{(2s)}$. Consider some copy j, match subset i with subset $\overrightarrow{i}$ in copy $\overrightarrow{j} + \overrightarrow{i}$. Similarly to Theorem

3.2.4, this implies that a face in some copy j has only one handle coming out of it. For each matching, a handle carrying four edges is attached between each pair of corresponding quadrilateral faces. In this manner, $4s^2$ links are completed, resulting in a quadrilateral embedding of $Q_{j+1}^{(2s)}$.

For a fixed j, pair off copy i of $Q_j^{(2s)}$ with copy $\overrightarrow{i} + \overrightarrow{j}$, for $1 \leq \overrightarrow{i} + \overrightarrow{j} \leq 2s$ (mod $2s$). For each such pairing, the two copies are linked together using $2^{2j-2}s^j$ handles. Therefore, there are $4(2^{2j-2}s^j)(2s) = 2^{2j+1}s^{j+1}$ quadrilateral faces on the handles between copy j and all other copies of opposite orientation. From each handle, choose one pair of opposite faces. The $2^{2j}s^{j+1}$ faces chosen are mutually vertex-disjoint and contain all the vertices of $Q_{j+1}^{(2s)}$. Letting j run from one to $2s$, we obtain $2s$ mutually disjoint sets of quadrilateral faces, satisfying $P(j+1)$.

The embedding obtained for $Q_{j+1}^{(2s)}$ is quadrilateral since the copies of $Q_j^{(2s)}$ are quadrilateral and the introduction of a handle that carries four edges only introduces quadrilateral faces. The number of faces $f_{n+1} = 4sf_n + \tilde{f}$, where $\tilde{f}$ is the number of new faces introduced by the handles. The number of handles is $(2^{2j-2}s^j)(4s^2) = 4^js^{j+2}$ since there are $4s^2$ links in total. Thus, $f_{n+1} = 4s(j2^{2j-1}s^{j+1}) + 2(4^js^{j+2}) = (j+1)2^{2j+1}s^{j+2}$. This shows that $P(j+1)$ follows from $P(j)$ for all $j \geq 1$. By Proposition 2.1.3,

$$\begin{aligned} g(Q_j^{(2s)}) &= 1 + \frac{j2^{2j}s^{j+1}}{4} - \frac{4^js^j}{2} \\ &= 1 + 2^{2j-2}s^j(sj-2) \end{aligned}$$

□

Since Theorem 3.2.4 includes the cases when $s = 1$ and when $s = 3$, the construction given in Theorem 3.5.4 can be slightly modified to obtain a formula for the genus of $Q_j^{(s)}$.

Theorem 3.5.5. *[28] The genus of $Q_j^{(s)}$ is given by*

$$g(Q_j^{(s)}) = 1 + 2^{j-3}s^j(j-4)$$

for s even and any positive integer j, or for $s = 1$ or $s = 3$ and for $j \geq 2$.

From Corollary 3.4.3, if $k = 2$, then, $Q_j^{(2)}$ is the $2j-$cube since $G_1 = C_4 = K_2 \square K_2$ and the following result is obtained.

Corollary 3.5.6. *[28]* $g(Q_{2j}) = 1 + 2^{2j-2}(j-2)$.

Using Theorem 3.5.4, we provide a construction for a quadrilateral embedding of $Q_i^{(2r)} \square C_{2s}$.

Theorem 3.5.7. *The genus of* $Q_i^{(2r)} \square C_{2s}$ *is given by*

$$g(Q_i^{(2r)} \square C_{2s}) = 1 + 2^{2i-1} s r^i (ir - 1).$$

Proof. First, minimally embed $2s$ copies of $Q_i^{(2r)}$ as described by Theorem 3.5.4. We assign s many copies one orientation and the other s copies the reverse orientation. This partition corresponds to the vertex-set partition of C_{2s}. From each copy of $Q_i^{(2r)}$, two links are to be made, both to copies of reverse orientation. For each link, $2^{2i}r^i$ edges are to be added and these are passed over $2^{2i-2}r^i$ handles with each handle carrying four edges. By Theorem 3.5.4, each copy of $Q_i^{(2r)}$ has $2r$ disjoint sets of $2^{2i-2}r^i$ mutually vertex-disjoint quadrilateral faces that cover all $2^{2i}r^i$ vertices of $Q_i^{(2r)}$. Since only two links are to be made from each copy of $Q_i^{(2r)}$, we choose any two of the available sets. The number of quadrilateral faces in each set is precisely the amount required to perform a link, as previously described. Since the embedding of each copy of $Q_i^{(2r)}$ is quadrilateral and each insertion of a handle introduces more quadrilateral faces, the obtained embedding of $Q_i^{(2r)} \square C_{2s}$ is quadrilateral. Applying Proposition 2.1.3, we get

$$\begin{aligned} g(Q_i^{(2r)} \square C_{2s}) &= 1 + \frac{2sir^{i+1}2^{2i} + 2sr^i 2^{2i}}{4} - \frac{2sr^i 2^{2i}}{2} \\ &= 1 + 2^{2i-1} s r^i (ir - 1). \end{aligned}$$

□

The following theorem generalises Theorem 3.5.7 by considering the repeated Cartesian product of even cycles.

Theorem 3.5.8. *The genus of* $Q_i^{(2r)} \square G_j$ *is given by*

$$g(Q_i^{(2r)} \square G_j) = 1 + M^{(j)} 2^{2i+j-2} r^i (j + ir - 2).$$

Proof. We let G denote $Q_i^{(2r)} \square G_j$. The proof proceeds by induction on j, where the property $P(j)$ is as follows: There is a quadrilateral embedding of G, with the number of faces $f_j = r^i 2^{2i+j-1} M^{(j)}(j + ir)$, including two disjoint sets each of $r^i 2^{2i+j-2} M^{(j)}$ mutually vertex-disjoint quadrilateral faces each such that each set contains all the $r^i 2^{2i+j} M^{(j)}$ vertices of G.

For $j = 1$, the quadrilateral embedding of $Q_i^{(2r)} \square C_{2m_1}$ is given by Theorem 3.5.7. The number of faces f_1 is given by $f_1 = \frac{1}{2} \left| E(Q_i^{(2r)} \square C_{2m_1}) \right|$, and thus

$$\begin{aligned} f_1 &= \frac{(2m_1)(2^{2i}r^i) + (2m_1)(i2^{2i}r^{i+1})}{2} \\ &= m_1 2^{2i} r^i + i m_1 2^{2i} r^{i+1} \\ &= 2^{2i} m_1 r^i (1 + ir). \end{aligned}$$

which conforms to the number of expected faces as stated by $P(1)$. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links, and the second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, each disjoint set would contain $2m_1(2^{2i-2}r^i) = 2^{2i-1}m_1 r^i$ quadrilateral faces. This follows since half of the $2m_1$ links are chosen, where each link has $2^{2i-2}r^i$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of $Q_i^{(2r)} \square C_{2m_1}$. Therefore, $P(1)$ holds.

Assuming $P(j)$ is true, we will show that $P(j+1)$ also holds. The graph we are considering is $G' = Q_i^{(2r)} \square G_{j+1}$. First, minimally embed $2m_{j+1}$ copies of G as described by $P(j)$. Assign m_{j+1} many copies one orientation and the other m_{j+1} copies the reverse orientation. This partition corresponds to the vertex-set partition of $C_{2m_{j+1}}$. From each copy of G, two links are to be made, both to copies of reverse orientation. The number of vertices of G is $r^i 2^{2i+j} M^{(j)}$, and thus we need to pass that many edges over a link joining two copies of G with different orientations. Therefore, the number of handles required for each link is $\frac{1}{4}\left(r^i 2^{2i+j} M^{(j)}\right)$ since each handle carries four edges. Using $P(j)$, we have two disjoint sets each of $r^i 2^{2i+j-2} M^{(j)}$ mutually vertex-disjoint quadrilateral faces and thus we have a sufficient number of faces such that we can construct these links using $r^i 2^{2i+j-2} M^{(j)}$ handles, where each handle carries four edges so that each new face formed is a quadrilateral. In this manner, the graph G' is embedded after constructing the $2m_{j+1}$ links.

The number of faces f_{j+1} in G' is $f_{j+1} = 2m_{j+1}f_j + \tilde{f}$, where $\tilde{f}$ denotes the number of new faces introduced by the handles. Note that $\tilde{f} = 4m_{j+1}(r^i 2^{2i+j-2} M^{(j)})$ since we have $2m_{j+1}$ links, with each link requiring $r^i 2^{2i+j-2} M^{(j)}$ handles and the difference in the number of faces is two per handle. Therefore,

$$\begin{aligned} f_{j+1} &= 2m_{j+1}(r^i 2^{2i+j-1} M^{(j)}(j+ir)) + 4m_{j+1}(r^i 2^{2i+j-2} M^{(j)}) \\ &= r^i 2^{2i+j} M^{(j+1)}(j+ir) + r^i 2^{2i+j} M^{(j+1)} \\ &= r^i 2^{2i+j} M^{(j+1)}(j+ir+1). \end{aligned}$$

This satisfies the expected number of faces as stated by $P(j+1)$. Next, we have to construct the two disjoint sets of faces. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links, and the second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, each disjoint set would contain $2m_{j+1}(r^i 2^{2i+j-2} M^{(j)}) = 2^{2i+j-1} r^i M^{(j+1)}$ quadrilateral faces. This follows since half of the $2s$ links are chosen, where each link has $r^i 2^{2i+j-2} M^{(j)}$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of G'. Therefore, $P(j+1)$ follows from $P(j)$ and since G is bipartite by Corollary 3.1.7, we calculate the genus using Proposition 2.1.3 to get

$$\begin{aligned} g(Q_i^{(2r)} \square\, G_j) &= 1 + \frac{r^i 2^{2i+j} M^{(j)}(j+ir)}{4} - \frac{2^{2i+j} r^i M^{(j)}}{2} \\ &= 1 + M^{(j)} 2^{2i+j-2} r^i (j+ir-2). \end{aligned}$$

□

We now consider the family of graphs of the Cartesian product of the j-cube with the repeated Cartesian product of paths.

Using Theorem 3.5.7, we show that $Q_i^{(2r)} \square\, P_{2s}$ has a quadrilateral embedding, and consequently, we derive its genus.

Theorem 3.5.9. *The genus of $Q_i^{(2r)} \square\, P_{2s}$ is given by*

$$g(Q_i^{(2r)} \square\, P_{2s}) = 1 + 2^{2i-2} r^i (2s(ir-1)-1).$$

Proof. We let G denote $Q_i^{(2r)} \square P_{2s}$ and note that G is bipartite by Corollary 3.1.7. We start by first embedding $G' = Q_i^{(2r)} \square C_{2s}$ as given in Theorem 3.5.7. To obtain G, we only need to remove $4^i r^i$ edges from G' between a copy of $Q_i^{(2r)}$ with one orientation and another copy of reverse orientation. The only way these two copies are connected is through $2^{2i-2} r^i$ handles, therefore, we remove these $2^{2i-2} r^i$ handles and the edges on them. The faces we have reintroduced are quadrilateral, since the embedding of G' is quadrilateral, producing a quadrilateral embedding for G. By Proposition 2.1.3, the genus of G is given by,

$$\begin{aligned} g(Q_i^{(2r)} \square P_{2s}) &= 1 + 2^{2i-1} s r^i (ir - 1) - 2^{2i-2} r^i \\ &= 1 + 2^{2i-2} r^i (2s(ir-1) - 1). \end{aligned}$$

□

As we restrict the next result to paths on an even number of vertices, let $m^{(j)}$ denote $\sum_{k=1}^{j} \frac{1}{2m_k}$ whilst noting that for $k \in \{1, \ldots, j\}$, m_k is even, restricting the repeated Cartesian product H_j to paths on an even number of vertices only. Using Theorem 3.5.9, we establish the genus of $Q_i^{(2r)} \square H_j$.

Theorem 3.5.10. *The genus of $Q_i^{(2r)} \square H_j$ is given by*

$$g(Q_i^{(2r)} \square H_j) = 1 + 2^{2i+j-2} r^i M^{(j)} (ir + j - m^{(j)} - 2).$$

Proof. We let G denote $Q_i^{(2r)} \square H_j$. The proof proceeds by induction on j, where the property $P(j)$ is as follows: There is a quadrilateral embedding of G, with $f_j = 2^{2i+j-1} r^i M^{(j)} (j + ir - m^{(j)})$, including two disjoint sets of $r^i 2^{2i+j-2} M^{(j)}$ mutually vertex-disjoint quadrilateral faces each, such that each set contains all the $r^i 2^{2i+j} M^{(j)}$ vertices of G.

For $j = 1$, the quadrilateral embedding of $Q_i^{(2r)} \square P_{2m_1}$ is given by Theorem 3.5.9. The number of faces f_1 is given by $f_1 = \frac{1}{2} \left| E(Q_i^{(2r)} \square P_{2m_1}) \right|$, and thus

$$\begin{aligned} f_1 &= \frac{(2m_1 - 1)(2^{2i} r^i) + (2m_1)(i 2^{2i} r^{i+1})}{2} \\ &= 2^{2i-1} r^i (2m_1 - 1 + 2m_1 ir) \\ &= 2^{2i} m_1 r^i \left(1 + ir - \tfrac{1}{2m_1}\right). \end{aligned}$$

which conforms to the number of expected faces as stated by $P(1)$. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links, and the second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, each disjoint set would contain $2m_1(2^{2i-2}r^i) = 2^{2i-1}r^i m_1$ quadrilateral faces. This follows since from $2m_1 - 1$ links, $\lceil \frac{2m_1-1}{2} \rceil$ are chosen, where each link has $2^{2i-2}r^i$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of $Q_i^{(2r)} \square P_{2m_1}$, therefore, $P(1)$ holds.

Assuming $P(j)$ is true, we will show that $P(j+1)$ also holds. The graph we are considering is $G' = Q_i^{(2r)} \square H_{j+1}$. First, minimally embed $2m_{j+1}$ copies of G' as described by $P(j)$. Assign m_{j+1} many copies one orientation and the other m_{j+1} copies the reverse orientation. This partition corresponds to the vertex-set partition of $P_{2m_{j+1}}$. From each copy of common orientation, except the end copy, of G, two links are to be made, all to copies of reverse orientation. We can construct these links using $r^i 2^{2i+j-2} M^{(j)}$ handles, where each handle carries four edges so that each new face formed is a quadrilateral. In this manner, the graph G' is embedded after constructing the $2m_{j+1} - 1$ links.

The number of faces f_{j+1} in G' is $f_{j+1} = 2m_{j+1} f_j + \tilde{f}$, where $\tilde{f}$ denotes the number of new faces introduced by the handles. Note that $\tilde{f} = 2(2m_{j+1} - 1)(r^i 2^{2i+j-2} M^{(j)})$ since we have $2m_{j+1} - 1$ links, with each link requiring $r^i 2^{2i+j-2} M^{(j)}$ handles and the difference in the number of faces is two per handle. Therefore,

$$\begin{aligned} f_{j+1} &= 2m_{j+1}(2^{2i+j-1} r^i M^{(j)}(j + ir - m^{(j)})) + (2m_{j+1} - 1)(r^i 2^{2i+j-1} M^{(j)}) \\ &= r^i 2^{2i+j} M^{(j+1)} \left(ir + j - m^{(j)} + \frac{2m_{j+1} - 1}{2m_{j+1}} \right) \\ &= r^i 2^{2i+j} M^{(j+1)} (ir + j + 1 - m^{(j+1)}). \end{aligned}$$

This satisfies the expected number of faces as stated by $P(j+1)$. Next, we have to construct the two disjoint sets of faces. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links. The second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, the disjoint set would contain $2\left\lceil \frac{2m_{j+1}-1}{2} \right\rceil (r^i 2^{2i+j-2} M^{(j)}) =$

$m_{j+1}2^{2i+j-1}r^iM^{(j)} = 2^{2i+j-1}r^iM^{(j+1)}$ quadrilateral faces. This follows since half of the $2m_{j+1}-1$ links are chosen, where each link has $r^i2^{2i+j-2}M^{(j)}$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of G'. Therefore, $P(j+1)$ follows from $P(j)$ and since G is bipartite by Corollary 3.1.7, we calculate the genus using Proposition 2.1.3 to get

$$\begin{aligned} g(Q_i^{(2r)}\Box H_j) &= 1 + \frac{r^i2^{2i+j}M^{(j)}(j+ir-m^{(j)})}{4} - \frac{2^{2i+j}r^iM^{(j)}}{2} \\ &= 1 + r^i2^{2i+j-2}M^{(j)}(j+ir-m^{(j)}) - 2^{2i+j-1}r^iM^{(j)}. \\ &= 1 + 2^{2i+j-2}r^iM^{(j)}(ir+j-m^{(j)}-2). \end{aligned}$$

□

We note that even though the path is not a regular graph, the graphs of the form $Q_i^{(2r)}\Box H_j$ also have a quadrilateral embedding.

3.6 | Genus of $S_3\Box P_{r+1}$

In this section, we consider another family of bipartite graphs, namely the the Cartesian product of the star on three vertices with a path.

Definition 3.6.1. *The* Cartesian product of the star $K_{1,s}$ and the path P_r, *denoted by $K_{1,s}\Box P_r$, is a graph on $(r+1)(s+1)$ vertices (u_i, w_j) for $0 \le i \le s$ and $0 \le j \le r$. Its $(s+r+2sr)$ edges are made up of adjacent pairs of vertices as follows,*

1. *(u_0, w_j) and (u_i, w_j) for $1 \le i \le s$ and $0 \le j \le r$ (forming the stars), and*

2. *(u_i, w_j) and (u_i, w_{j+1}) for $0 \le i \le s$ and $0 \le j \le r-1$ (forming the paths).*

The result stated hereunder which determines a lower bound for the orientable genus of $K_{1,3}\Box P_{r+1}$ is also new and was derived while pursuing the crossing number of $K_{1,3}\Box P_{r+1}$ on any surface S. The corresponding paper is in its final writing stages.

Lemma 3.6.2. *For $r \geq 1$, the orientable genus of $K_{1,3}\square P_{r+1}$ is given by*

$$g(K_{1,3}\square P_{r+1}) \geq \left\lceil \frac{r-1}{2} \right\rceil.$$

Proof. The proof proceeds by induction on r. We split the base case into odd and even cases. For the odd base case, $K_{1,3}\square P_2$ is a planar graph, thus $g(K_{1,3}\square P_2) = 0$, satisfying that $g(K_{1,3}\square P_2) \geq \lceil \frac{1-1}{2} \rceil = 0$. For the even base case, we note that $K_{1,3}\square P_3$ contains a subgraph H homeomorphic to $K_{3,3}$, thus $g(K_{1,3}\square P_3) \geq 1$. This is illustrated in Figure 3.10, where the coloured vertices represent the vertices of H, which is obtained by deleting edges e_1 and e_2. The embeddings for the base cases are found in Figure 3.10.

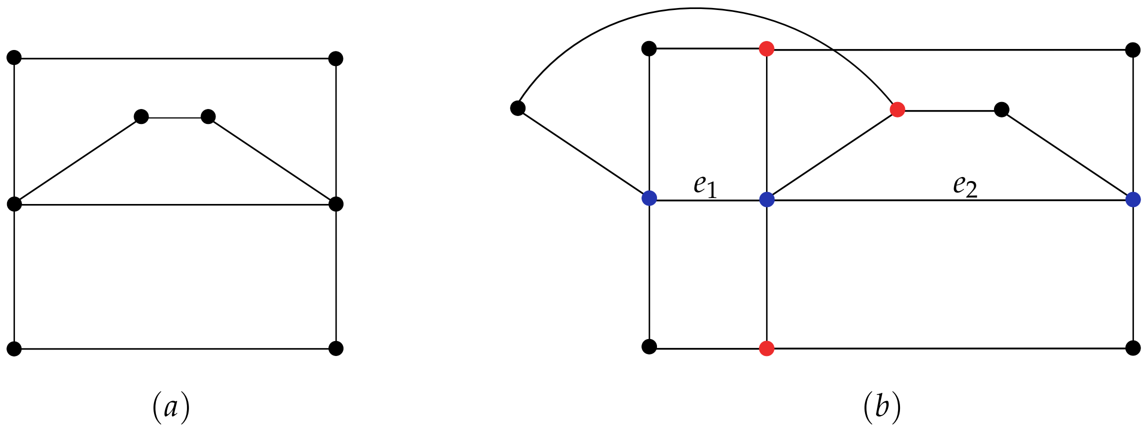


Figure 3.10: $(a) : K_{1,3}\square P_2$, $(b) : K_{1,3}\square P_3$ with $K_{3,3}$ homeomorph

The induction hypothesis states that $g(K_{1,3}\square P_h) \geq \lceil \frac{h-2}{2} \rceil$ for all $h \leq 2k$. We need to show that $g(K_{1,3}\square P_{2k+1}) \geq k$ and $g(K_{1,3}\square P_{2k+2}) \geq k$. Suppose that $g(K_{1,3}\square P_{2k+1}) < k$. Thus, $k-1 \geq g(K_{1,3}\square P_{2k+1}) \geq g(K_{1,3}\square P_{2k}) = k$, by the inductive hypothesis, which is a contradiction. Similarly, suppose that $g(K_{1,3}\square P_{2k+2}) < k$. Then $k-1 \geq g(K_{1,3}\square P_{2k+2}) \geq g(K_{1,3}\square P_{2k+1}) \geq k$, by the result above, which is a contradiction.

Therefore, $g(K_{1,3}\square P_{2k+1}) \geq k$ and $g(K_{1,3}\square P_{2k+2}) \geq k$ as required. $\square$

4

Genus of several direct product graphs

In this chapter, we consider the direct product of several bipartite graphs. Focusing on the direct product of two bipartite graphs, the results of this chapter are summarised in the theorem below.

Theorem 4.0.1. $g(G \times H) = \begin{cases} 2(4(pq)^2 - (p+q)^2 + 1), & \textit{if } G = K_{2p,2q} \textit{ and } H = K_{2p,2q}, \\ 2, & \textit{if } G = C_{2r} \textit{ and } H = C_{2s}, \\ 2^{j_1+j_2-3}(j_1 j_2 - 4) + 2, & \textit{if } G = Q_{j_1} \textit{ and } H = Q_{j_2}. \end{cases}$

4.1 | General Results

First, we introduce some introductory results on the direct product of two graphs which we will use in the proofs for this chapter.

Lemma 4.1.1. *Let G and H be two connected graphs on n_1 and n_2 vertices respectively with m_1 and m_2 edges, respectively. The direct product $G \times H$ has $n = n_1 n_2$ vertices and the number of edges m is $2m_1 m_2$. Moreover, a vertex (x, y) in $G \times H$ has degree $deg_G(x)deg_H(y)$.*

Proof. Let G and H be two connected graphs on n_1 and n_2 vertices respectively. By definition, the vertex-set of $G \times H$ is $V(G) \times V(H)$, thus, the number of vertices is $n_1 n_2$.

Let $(x, y) \in V(G \times H)$ whose degree is denoted by $deg_{G\times H}((x, y))$. Let (u, v) be another vertex in $G \times H$. Then, (x, y) and (u, v) are adjacent to each other in $G \times H$ if and only if x is adjacent to u in G and y is adjacent to v in H. Fix the vertices x and u in G. Then, (x, y) is adjacent to (u, v) for any $v \in V(H)$ such that y and v are adjacent in H. The vertex y has precisely $deg_H(y)$ vertices adjacent to it in H, thus, (x, y) is adjacent to $deg_H(y)$-many vertices of the form (u, v) in $G \times H$. For each one of these adjacencies, there are $deg_G(x)$ vertices adjacent to x in G. Thus, $deg_{G\times H}((x, y)) = deg_G(x)deg_H(y)$.

We calculate the number of edges by the Handshaking Lemma in Lemma 1.1.1 as follows.

$$\begin{aligned}
2m &= \sum_{(v_1,v_2)\in V(G\times H)} deg((v_1, v_2)) \\
&= \sum_{v_1\in V(G)} deg(v_1) \times \sum_{v_2\in V(H)} deg(v_2) \\
&= 2|E(G)|.2|E(H)| \text{ by applying the Handshaking Lemma to } G \text{ and } H \\
&= 4m_1m_2 \\
\Rightarrow m &= 2m_1m_2.
\end{aligned}$$

□

By simple induction, we can extend the result to the direct product of r graphs as follows. We shall denote the direct product of r graphs $G_1, G_2, \ldots, G_r$ by $\bigtimes_{i=1}^{r} G_i$.

Corollary 4.1.2. *Let $G_1, G_2, \ldots, G_r$ be connected graphs and let $G = \bigtimes_{i=1}^{r} G_i$. Then, the number of edges of G is given by $2^{r-1}\prod_{i=1}^{r}|E(G_i)|$. Moreover, for a vertex $v = (v_1, v_2, \ldots, v_r) \in V(G)$, the degree of v is given by*

$$deg_G(v) = deg_{G_1}(v_1).deg_{G_2}(v_2).\ldots.deg_{G_r}(v_r).$$

We state more elementary results on the direct product of two graphs, however, we omit the proofs of these theorems because, even though they are not difficult to present, they are not in the scope of this study. The following theorem characterises connectedness for the direct product of two graphs.

Theorem 4.1.3. *[13, 26] The direct product of two connected graphs is connected if and only if at least one of them has an odd cycle.*

Next, we state a characterisation of bipartiteness for the direct product of two graphs.

Theorem 4.1.4. *[26] The direct product of two graphs is bipartite if and only if at least one of them is bipartite.*

In this chapter, we will be considering bipartite graphs and their direct products. Thus, these families of graphs do not have odd cycles and so, their direct product is disconnected. More importantly, when two graphs do not have any odd cycles, then their direct product will have exactly two components, as stated in the result below.

Corollary 4.1.5. *[13, 26] If G and H are two connected graphs with no odd cycles, then $G \times H$ has exactly two connected components.*

A *1-factor* in a graph G is a 1-regular spanning subgraph of G, thus, the edge-set of a 1-factor is a perfect matching in G. The following definition allows us to establish an embedding for some families of direct products of bipartite graphs.

Definition 4.1.6. *[31] Let G be a connected bipartite graph such that $V(G) = V_1 \cup V_2$ and each edge has one vertex from V_1 and the other from V_2. We say that G has a* diagonalizable quadrilateral embedding *(DQE) in some surface S if:*

1. *$G \to S$ is a quadrilateral embedding,*
2. *there exists a graph $G' = (V, E')$ having an embedding $G' \to S$ and a 1-factor F such that both vertices of each edge of F are from V_1 or from V_2 and $(G' \to S) - F = G \to S$.*

If G is disconnected, then it has a DQE if each component has a DQE.

From now onwards, the edges of F will be referred to as diagonals.

From Corollary 4.1.5, it is easy to see that several of the results in this chapter will include disconnected graphs. Let G be a graph on n vertices and m edges with k components labelled $C_1, C_2, \ldots, C_k$, each with n_i vertices and m_i

edges, for $i \in \{1, \ldots, k\}$. If G has a DQE, then each component has a quadrilateral embedding on some surface S, therefore, the genus of each component is $g(C_i) = \frac{m_i}{4} - \frac{n_i}{2} + 1$, for $i \in \{1, \ldots, k\}$ by Proposition 2.1.3. As a consequence of this and Corollary 1.2.12 stating that the genus of a graph with k components is the sum of the genera of its components, we get that

$$\begin{aligned} g(G) &= \sum_{i=1}^{k} g(C_i) \\ &= \sum_{i=1}^{k} \left(\frac{m_i}{4} - \frac{n_i}{2} + 1 \right) \\ &= \frac{m_1 + m_2 + \ldots + m_k}{4} - \frac{n_1 + n_2 + \ldots + n_k}{2} + k \\ &= \frac{m}{4} - \frac{n}{2} + k. \end{aligned} \tag{4.1}$$

We will often employ this formula when calculating the genus of the direct product of two (or more) bipartite graphs. Another definition which will be used in the main theorem of this section is that of a generalised embedding scheme as given by Stahl [23]. For this purpose, we first define a voltage map.

Definition 4.1.7. *Let $\mathcal{B}$ be a finite group and let G be a graph whose edges have been assigned plus and minus orientations. Then the function $\lambda : E(G) \to \mathcal{B}$ which assigns a value from $\mathcal{B}$ to each edge of G is called a* voltage map.

Using this definition, we can define a generalised embedding scheme.

Definition 4.1.8. *A* generalised embedding scheme (P, λ) *for the graph G, consists of a rotation system P and a voltage map λ on G with values in $\mathbb{Z}_2$.*

Next, we define what it means for a graph to be irreducible.

Definition 4.1.9. *For a graph G to be* irreducible, *it is either a single vertex or every vertex has degree at least three.*

Another useful definition when considering generalised embedding schemes is the number of one's that are assigned to the edges of a cycle in a graph.

Definition 4.1.10. *Let G be a graph with generalised embedding scheme (P, λ) having a cycle C. C is said to be λ-*trivial *if it contains an even number of edges $e_1, e_2, \ldots, e_k$ such that for all $i \in \{1, \ldots, k\}$, $\lambda(e_i) = 1$.*

As a consequence, Stahl found a characterisation relating generalised embedding schemes with orientable embeddings of graphs, given in the theorem below.

Theorem 4.1.11. *[23] If G is an irreducible graph, then the embedding scheme (P, λ) defines an orientable embedding of G on a surface S if and only if every cycle of G is λ-trivial.*

The first general result shows what conditions are necessary for the direct product of two graphs to have a DQE.

Theorem 4.1.12. *[31] Let G and H be graphs, with G being a bipartite graph. If G has a diagonalizable quadrilateral embedding, then the direct product $G \times H$ also has a diagonalizable quadrilateral embedding.*

Proof. Here, we prove a stronger result, namely that if $f \in F$ is a diagonal joining vertices x and y from G, then there exists a diagonal f_u joining (x, u) to (y, u) from $G \times H$, for each $u \in V(H)$.

For two vertex rotations $P_1 = (p_1, \ldots, p_r)$ and $P_2 = (q_1, \ldots, q_k)$, we define $P_1 \cup P_2$ to be the permutation $(p_1, \ldots, p_r, q_1, \ldots, q_k)$, we denote by $P_1 \times \{u\}$ the permutation $((p_1, u), \ldots, (p_r, u))$. For brevity, we shall write (a, b) instead of $((x, y), (a, b))$ in P_{xy} and similarly, we shall write b instead of (x, b) in Q_x.

The proof proceeds by induction on the number of edges m of H.

For the base case, we consider the case when $m = 1$, that is, $H = K_2$. Label the vertices of H by u and v. Since both G and H are bipartite graphs, then by Theorem 4.1.4 $G \times H$ is also bipartite, therefore, by Corollary 4.1.5, $G \times H$ has exactly two components. It turns out that both of these components are isomorphic to G since a vertex (x, w) in $G \times H$ is adjacent to (y, z) if and only if x is adjacent to y in G and w is adjacent to z in H. However, H has only one edge so, without loss of generality, we let $w = u$ and $z = v$. In this way, the first co-ordinate of the vertices adjacent to each other in $G \times H$ reflect the adjacencies in G, whilst the second co-ordinate alternates between u and v, making each component isomorphic to G.

Let (Q, λ) be an embedding scheme which describes a DQE for G in some surface S. Using this embedding scheme, we define the rotations and voltage assignments for $G \times K_2$ as follows. For each $x \in V(G)$, let Q_x denote the rotation at x, then, $Q'_{xu} = Q_x \times \{v\}$ and $Q'_{xv} = Q_x \times \{u\}$. Also, the voltage map λ' is defined by $\lambda'(\{(x,u),(y,v)\}) = \lambda'(\{(x,v),(y,u)\}) = \lambda(\{x,y\})$ for each $xy \in E(G)$. We can see that $Q' = \{Q'_{xz} : x \in V(G), z \in V(K_2)\}$ describes a quadrilateral embedding for the two components of $G \times K_2$. Additionally, if a diagonal f joins vertices x and y in G, then we can choose diagonals in $G \times K_2$ such that f_z joins (x,z) with (y,z), for $z \in V(K_2)$. Therefore, (Q', λ') represents a DQE.

Now suppose that the result holds for any graph having m edges and consider the graph H that has $m+1$ edges. Let $e = uv$ be an edge of H such that $deg_H(v) > 1$ and $H' = H - e$ is connected. By the induction hypothesis, the graph $G \times H'$ has a DQE on some surface S. We represent the embedding by an embedding scheme (P, λ_2) where $P = \{P_{xy} : x \in V(G), y \in V(H')\}$. We also consider an embedding scheme (Q, λ) that describes a DQE for G. Let x, a, y, b be a quadrilateral face in G such that the vertices x and y are connected by a diagonal f_1 situated inside the face. Then, in $G \times H'$, the vertices (x,v) and (y,v) are to be connected through a diagonal f'_{1v} and similarly, these vertices are to be connected by a diagonal f_{1v} in $G \times K_2$, where $E(K_2) = e = uv$. Therefore, the rotations mentioned for G have the following form:

$$\begin{aligned} Q_x &= (a\ c\ \ldots\ b), \\ Q_y &= (b\ d\ \ldots a), \\ Q_w &= (w_1\ w_2\ \ldots w_t) \text{ for } w \in V(G) \setminus \{x,y\} \end{aligned} \tag{4.2}$$

Meanwhile, the rotations given by P have the following form:

$$\begin{aligned} P_{xv} &= ((x_1, v_1)\ (x_2, v_2)\ \ldots\ (x_n, v_r)), \\ P_{yv} &= ((y_1, v'_1)\ (y_2, v'_2)\ \ldots\ (y_n, v'_s)), \\ P_{x'w'} &= ((x'_1, w'_1)\ (x'_2, w'_2)\ \ldots\ (x'_n, w'_k)) \text{ otherwise.} \end{aligned} \tag{4.3}$$

We now define the rotations for $G \times H$ from Equation (4.2) and Equation (4.3) as follows:

$$\begin{aligned}
P'_{xv} &= P_{xv} \cup (Q_x \times \{u\}), \\
P'_{yv} &= P_{yv} \cup (Q_y \times \{u\}), \\
P'_{xu} &= Q_x \times \{v\}, \\
P'_{yu} &= Q_y \times \{v\}, \\
P'_{wv} &= Q_w \times \{u\}, \text{ for } w \in V(G) \setminus \{x,y\} \\
P'_{wu} &= Q_w \times \{v\}, \text{ for } w \in V(G) \setminus \{x,y\} \\
P'_{x'w'} &= P_{x'w'}, \text{ otherwise.}
\end{aligned} \tag{4.4}$$

For an edge e in $G \times H$ that is also in $G \times H'$, its label $\lambda'(e) = \lambda_2(e)$, whilst for an edge $e = \{(x,v),(y,u)\}$ for some $xy \in E(G)$, $\lambda'(e) = \lambda(xy)$.

If the degree of u is one, then we are done. If not, we simply interchange u with v and repeat the process above. By repeating this process for each of the diagonals $f_2, \ldots, f_m$ we obtain a quadrilateral embedding of the graph $G \times H$ represented by (P', λ'). The final embedding is orientable by taking the labelling in λ to be zero everywhere, and thus, λ' constantly labels every edge by zero. Since any cycles in the graph have zero edges labelled by one, then, there is an even number of edges e such that $\lambda'(e) = 1$. The result follows by Theorem 4.1.11. $\square$

By $M^{(r)}$, we denote the product $\prod_{i=1}^{r} |E(G_i)|$ and similarly, $N^{(r)}$ denotes the product $\prod_{i=1}^{r} |V(G_i)|$. Next, we establish a formula for the genus of this product when a number of these graphs are bipartite.

Theorem 4.1.13. *[31] Let the connected bipartite graph G_1 have a quadrilateral embedding. Let $G_2, \ldots, G_r$ be connected graphs, k of them bipartite. Then, the genus of $\bigtimes_{i=1}^{r} G_i$ is given by,*

$$g\left(\bigtimes_{i=1}^{r} G_i\right) = 2^{r-3}.M^{(r)} - \frac{N^{(r)}}{2} + 2^k.$$

Proof. Since G_1 is a bipartite graph with a quadrilateral embedding, then it has an even number of vertices, which allows it to have a diagonalisable quadrilateral embedding by adding edges between vertices in the same partite set in such a way that they are pairwise disjoint and cover all the vertices of G_1. Applying Theorem 4.1.12 $r-1$ times, we get a DQE for $G_1 \times \cdots \times G_r$. The number of vertices of $G_1 \times \cdots \times G_r$ is $N^{(r)}$, the number of edges is $2^{r-1}M^{(r)}$ by Corollary 4.1.2 and the number of components is 2^k by Theorem 4.1.4 and Corollary 4.1.5. Substituting these values in Equation (4.1), we get that $g\left(\bigtimes_{i=1}^{r} G_i\right) = 2^{r-3}.M^{(r)} - \frac{N^{(r)}}{2} + 2^k$. □

4.2 | Genus of $\bigtimes_{i=1,j=1}^{k} K_{2p_i,2q_j}$

We first find an orientable DQE for $K_{2p,2q}$, then, we take the direct product of several complete bipartite graphs and establish the genus.

Theorem 4.2.1. *[31] The graph $K_{2p,2q}$ has an orientable DQE for each $p \geq 1, q \geq 1$.*

Proof. We first construct a quadrilateral embedding of $K_{2p,2q}$ given in [21, 28], also found in Remark 2.1.4 and Lemma 2.1.5. Then, for $i = 1, 3, 5, \ldots, 2p-1$, we join the vertices i and $i+1$ in the quadrilateral $i, 2p+1, i+1, 2q$. We also join the vertices j and $j+1$, for $j = 2p+1, 2p+2, \ldots, 2q-1$ in the quadrilateral $1, j, 2p, j+1$. The diagonals cover all the vertices of $K_{2p,2q}$ and are pairwise disjoint. Thus, we have found a suitable orientable DQE. □

Next, we employ Theorem 4.1.12, Theorem 4.1.13 and Theorem 4.2.1 to establish the genus of the direct product of several complete bipartite graphs. For the direct product of complete bipartite graphs $K_{2p_1,2q_1} \times \bigtimes_{i=2}^{k} K_{p_i,q_i}$, we let $M^{(k)}$ denote the product $\prod_{i=1}^{k} p_i q_i$ and $N^{(k)}$ denote the product $\prod_{i=1}^{k}(p_i + q_i)$.

Corollary 4.2.2. *[31] The direct product of $K_{2p_1,2q_1}$ with other complete bipartite graphs K_{p_i,q_i}, for $i = 2, \ldots, k$, the genus of $G = K_{2p_1,2q_1} \times \bigtimes_{i=2}^{k} K_{p_i,q_i}$ is given by,*

$$g(G) = 2^{k-1}(M^{(k)} + 1) - N^{(k)}.$$

Proof. By Theorem 4.2.1, $K_{2p_1,2q_1}$ has an orientable DQE, therefore, by applying Theorem 4.1.12 $k-1$ times we get that G also has a DQE. The number n of vertices of G is $n = (2p_1 + 2q_1)(p_2 + q_2)\ldots(p_k + q_k) = 2N^{(k)}$. The number of edges m of G, given by Corollary 4.1.2, is $2^{k-1}(4p_1q_1 \prod_{i=2}^{k} p_i q_i) = 2^{k+1}M^{(k)}$.

By Theorem 4.1.13 the genus is given by,

$$g(G) = \frac{2^{k+1}M^{(k)}}{4} - \frac{2N^{(k)}}{2} + 2^{k-1} = 2^{k-1}(M^{(k)} + 1) - N^{(k)}.$$

□

For the special case when all the complete bipartite graphs have even partite sets, the number of vertices is given by $n = 2^k N^{(k)}$, the number of edges is given by $m = 2^{3k-1}(pq)^k$ and the number of components remains 2^{k-1}. Thus, by substituting these values in Theorem 4.1.13, the genus of $\bigtimes_{i=1}^{k} K_{2p,2q}$ is given by the formula below.

Corollary 4.2.3. *[31] The genus of $\bigtimes_{i=1}^{k} K_{2p,2q}$, for $p \geq 1$ and $q \geq 1$ is given by,*

$$g\left(\bigtimes_{i=1}^{k} K_{2p,2q}\right) = 2^{k-1}(2^{2k-2}(pq)^k - (p+q)^k + 1).$$

4.3 | Genus of $\bigtimes_{i=1}^{k} C_{r_i}$

In this section, we start by establishing a diagonalisable quadrilateral embedding of the direct product of two cycles C_r and C_s. Then, we show that the genus of the direct product of two even cycles is two whilst the genus of two cycles is one otherwise. By proving these two results, we then establish the genus of the direct product of several cycles.

We describe a diagonalisable quadrilateral embedding of the direct product of two cycles C_r and C_s.

Theorem 4.3.1. *[31] The direct product of two cycles C_r and C_s has a quadrilateral embedding. Moreover, this embedding is a DQE if and only if at least one of the cycles is an even cycle.*

Proof. Denote the vertices of the first circuit by $1, \ldots, r$ and those of the second circuit by $1, \ldots, s$ in the cyclic ordering. For $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$, define the rotations at a vertex (i, j) of $C_r \times C_s$ as follows:

$$P_{ij} = ((i-1, j-1)\ (i-1, j+1)\ (i+1, j+1)\ (i+1, j-1)). \tag{4.5}$$

Then, let $P : E(C_r \times C_s) \to E(C_r \times C_s)$ be defined by $P((x,y)(u,v)) = (u,v)P_{uv}((x,y))$. Then, the orbits under P, given in 4.6, correspond to quadrilateral faces of the embedding of $C_r \times C_s$. The collection of permutations $P_{11}, P_{12}, \dots, P_{rs}$ as defined in Equation (4.5) describes a quadrilateral embedding for each component of $C_r \times C_s$.

$$\{(i,j), (i+1, j+1), (i+2, j), (i+1, j-1)\}. \tag{4.6}$$

We have yet to show that this quadrilateral embedding can exhibit a DQE. If both r and s are odd, then the number of vertices of $C_r \times C_s$ is odd, therefore, it cannot have a DQE. Hence, suppose r is even. The vertices (i,j) and $(i+2,j)$ can be connected by a diagonal for $i = 4k+1$ and $i = 4k+2$, for $k = 0,1,2,\dots$ and $j = 1,\dots,s$. In this way, all the diagonals are pairwise disjoint and all the vertices of $C_r \times C_s$ are covered. □

Next, we introduce a result by Farzan and Waller [8] about the planarity of the direct product of two cycles.

Theorem 4.3.2. *[8] The direct product of any two cycles is non-planar.*

The proof of this theorem is omitted. Using this result, we can establish the genus of $C_r \times C_s$ as follows.

Corollary 4.3.3. *[31] The genus of the direct product of two cycles C_r and C_s is given by,*

$$g(C_r \times C_s) = \begin{cases} 2, & \text{if } r \text{ is even and } s \text{ is even,} \\ 1, & \text{otherwise .} \end{cases}$$

Proof. By Theorem 4.1.3, the direct product of two cycles is connected if and only if one of them is an odd cycle. If both r and s are even, then, by Corollary 4.1.5, $C_r \times C_s$ has exactly two components. Therefore, by Theorem 4.3.2 and Theorem 1.2.10, $g(C_r \times C_s) \geq 2$. All that is left to show is that each component of $C_r \times C_s$ can be embedded on the torus, or equivalently, S_1. First, embed $C_r \times C_s$ using the quadrilateral embedding given in Theorem 4.3.1. The number of faces f_K of any component K in this embedding is $\frac{m_K}{2}$, where m_K denotes the number of edges in K. Every vertex in the component has degree four since all the vertices in $C_r \times C_s$ have degree four, and if some vertex v in K has a smaller degree, then it must be connected to some vertex in another component, meaning that this vertex should also be in K. By the Handshaking Lemma, the number of edges in a component is twice the number of vertices n_K in that component. Finally, by Corollary 1.2.8 we get,

$$\begin{aligned}
2 - 2g(K) &= n_K - m_K + f_K \\
\Rightarrow 2g(K) &= 2 - n_K + m_K - f_K \\
&= 2 - \frac{m_K}{2} + m_K - \frac{m_K}{2} \\
&= 2 \\
\Rightarrow g(K) &= 1.
\end{aligned}$$

Therefore, both components of $C_r \times C_s$ have genus one and by Theorem 1.2.10, $g(C_r \times C_S) = 2$. Meanwhile, if either r or s is odd, $C_r \times C_s$ is connected and by Proposition 2.1.3, $g(C_r \times C_s) = 1 + \frac{2rs}{4} - \frac{rs}{2} = 1$. □

As a result of Corollary 4.3.3, we proceed by establishing the genus of the direct product of several cycles. Let $N^{(k)}$ be the product $\prod_{i=1}^{k} r_i$.

Corollary 4.3.4. *[31] Let G be the graph $C_{2r_1} \times \bigtimes_{i=2}^{k} C_{r_i}$ and let s be the number of even cycles among C_{r_i} for $i \in \{2, \ldots, k\}$, that is, $s = |\{r_i : 2 \leq i \leq k, r_i \text{ even}\}|$. Then the genus of G is given by,*

$$g(G) = N^{(k)}(2^{k-2} - 1) + 2^s.$$

Proof. Firstly, the direct product $C_{2r_1} \times C_{r_2}$ has a DQE by Theorem 4.3.1. Therefore, G has a DQE by successively applying Theorem 4.1.12. The number of vertices of G is given by $2N^{(k)}$ whilst the number of edges is given by $m = 2^k N^{(k)}$. Substituting these values in Theorem 4.1.13, we get that,

$$\begin{aligned}
g(G) &= \frac{2^k N^{(k)}}{4} - \frac{2N^{(k)}}{2} + 2^s \\
&= N^{(k)}(2^{k-2} - 1) + 2^s.
\end{aligned}$$

□

As a special case of Corollary 4.3.4, we consider the case when $r_i = 2r$, for all $i \in \{1, \ldots, k\}$.

Corollary 4.3.5. *[31] The genus of $\times_{i=1}^{k} C_{2r}$ is given by,*

$$g\left(\bigtimes_{i=1}^{k} C_{2r}\right) = 2^{k-1}(r^k(2^{k-2} - 1) + 1).$$

4.4 | Genus of $\times_{i=1}^{k} Q_{j_i}$

Similarly to previous sections in this chapter, we first find a quadrilateral embedding of the j-cube Q_j, then show it has an orientable DQE. Finally, we take the direct product of multiple j-cubes and determine its genus.

Theorem 4.4.1. *[31] The graph Q_j has an orientable DQE whenever $j \geq 2$.*

Proof. We first start by constructing a quadrilateral embedding for Q_j, which is given in [1, 19, 28], also found in Theorem 3.5.5. The vertices of Q_j are labelled by some binary sequence $a_1 a_2 \ldots a_j$ where two vertices are adjacent to each other if and only if their sequences differ by exactly one bit. Note that Q_j is bipartite, with the partite sets being $A = \{a : a$ has an even number of 1's$\}$ and $B = \{b : b$ has an odd number of 1's$\}$.

Let a be some binary sequence of length $j - 3$. Then, inside the quadrilateral $0a00\ 0a01\ 1a01\ 1a00$, we can add the diagonal $0a00\ 1a01$ since these two vertices differ in exactly two places. Similarly, we add the diagonal $0a01\ 1a11$ inside the quadrilateral $0a01\ 0a11\ 1a11\ 1a01$, the diagonal $0a11\ 1a10$ inside $0a11\ 0a10\ 1a10\ 1a11$, and the diagonal $0a10\ 1a00$ inside $0a10\ 0a00\ 1a00\ 1a10$. Note that for any a, the diagonals listed above are pairwise disjoint.

Let S be the set of all binary sequences of length $j - 3$. Then, the set of diagonals

$$\bigcup_{a \in S} F_a = \bigcup_{a \in S} \{0a00\ 1a01, 0a01\ 1a11, 0a11\ 1a10, 0a10\ 1a00\}$$

covers all the vertices of Q_j, making $\bigcup_{a \in S} F_a$ a 1-factor. Also, note that each diagonal in F has either both vertices in A or both in B. Therefore, we have found a suitable DQE for Q_j. $\square$

As a result of Theorem 4.1.12, Theorem 4.1.13 and Theorem 4.4.1, we can establish the genus of the direct product of several j_i-cubes as follows. For the following result, let $J^{(k)} = \prod_{i=1}^{k} j_i$.

Corollary 4.4.2. *[31] For $i \in \{1, \ldots, k\}$, let $j_i \geq 2$. Let $j = j_1 + \ldots + j_k$. Then, the genus of $\bigtimes_{i=1}^{k} Q_{j_i}$ is given by,*

$$g\left(\bigtimes_{i=1}^{k} Q_{j_i}\right) = 2^{j-3}(J^{(k)} - 4) + 2^{k-1}.$$

Proof. For $i \in \{1, \ldots, k\}$, let $G_i = Q_{j_i}$. By Theorem 4.4.1, G_1 has a DQE, thus, by applying Theorem 4.1.12, $G_1 \times G_2$ also has a DQE. By applying Theorem 4.1.12 for $k-1$ times, we get that $\bigtimes_{i=1}^{k} Q_{j_i}$ has a DQE. The number of vertices in this product is given by 2^j whilst the number of edges m is given by $m = 2^{k-1} \prod_{i=1}^{k} 2^{j_i - 1} j_i = 2^{j-1} J^{(k)}$. Therefore, by Theorem 4.1.13 the genus of $\bigtimes_{i=1}^{k} Q_{j_i}$ is,

$$g\left(\bigtimes_{i=1}^{k} Q_{j_i}\right) = \frac{2^{j-1} J^{(k)}}{4} - \frac{2^j}{2} + 2^{k-1} = 2^{j-3}(J^{(k)} - 4) + 2^{k-1}.$$

□

If $j_i = r$ for all $i \in \{1, \ldots, k\}$, then, the genus of $\bigtimes_{i=1}^{k} Q_r$ is as follows.

Corollary 4.4.3. *[31] The genus of $\bigtimes_{i=1}^{k} Q_r$ is given by,*

$$g\left(\bigtimes_{i=1}^{k} Q_r\right) = 2^{rk-3}(r^k - 4) + 2^{k-1}.$$

5

Conclusion

5.1 | Summary of Results

Here we present the main results covered in this dissertation. The first family of bipartite graphs encountered was the complete bipartite graph $K_{p,q}$, whose genus is given by the following theorem.

Theorem 5.1.1. *[3, 21] The genus of the complete bipartite graph $K_{p,q}$ is given by*

$$g(K_{p,q}) = \left\lceil \frac{(p-2)(q-2)}{4} \right\rceil .$$

Chapters 3 and 4 respectively cover the Cartesian product and direct product of families of bipartite graphs. We shall group together some results on the two products of families of bipartite graphs in order to be able to easily compare the genus of each of the two product graphs. We begin by first considering the complete bipartite graph, whose partite sets are even.

Theorem 5.1.2. *[28, 31] The genus of the product of two complete bipartite graphs $K_{2p,2p}$ and $K_{2q,2q}$ is given by,*

$$g(K_{2p,2p} * K_{2q,2q}) = \begin{cases} 1 + 4pq(p+q-2), & \textit{if } * \textit{ denotes the Cartesian product,} \\ 2(4pq(pq-1)+1), & \textit{if } * \textit{ denotes the direct product.} \end{cases}$$

From this, we derived the genus of the repeated product of complete bipartite graphs as follows.

Theorem 5.1.3. *[28, 31] The genus of the repeated product of the complete bipartite graph $K_{2p,2p}$ for k times is given by,*

$$g\left(\ast K_{2p,2p}\right) = \begin{cases} 1 + 2^{2k-2}p^k(pk-2), & \textit{if } \ast \textit{ denotes the repeated Cartesian product,} \\ 2^{k-1}(2^{2k-2}p^{2k} - 2^k p^k + 1), & \textit{if } \ast \textit{ denotes the repeated direct product.} \end{cases}$$

Next, we focused on another well-known family of bipartite graphs, namely even cycles.

Theorem 5.1.4. *[28, 31] The genus of the repeated product of two even cycles C_{2r} and C_{2s} is given by,*

$$g(C_{2r} \ast C_{2s}) = \begin{cases} 1, & \textit{if } \ast \textit{ denotes the Cartesian product,} \\ 2, & \textit{if } \ast \textit{ denotes the direct product.} \end{cases}$$

From this we were able to establish the genus of the repeated product of cycles C_{2r_i}. Let $M^{(k)}$ denote $\prod_{i=1}^{k} r_i$.

Theorem 5.1.5. *[28, 31] The genus of the repeated product of k even cycles C_{2r_i} is given by,*

$$g\left(\ast C_{2r_i}\right) = \begin{cases} 1 + 2^{k-2}M^{(k)}(k-2), & \textit{if } \ast \textit{ denotes the repeated Cartesian product,} \\ 2^{k-1}M^{(k)}(2^{k-2}-1) + 2^{k-1}, & \textit{if } \ast \textit{ denotes the repeated direct product.} \end{cases}$$

Apart from these families of bipartite graphs, we have considered the Cartesian product of paths. The Cartesian product of j paths for $j \geq 2$, is defined recursively as follows,

$$H_1 = P_{m_1}$$

$$H_j = H_{j-1} \square P_{m_j}.$$

Whilst all paths are bipartite, we restrict the first three paths to be on an even number of vertices. Let $M^{(j)}$ be defined as before and let $m^{(i)} = \sum_{i=1}^{j} \frac{1}{m_i}$. The following theorem establishes the genus of H_j, where the first three paths in the product are all on an even number of vertices.

Theorem 5.1.6. *[28] The genus of H_j, for $j \geq 3$ and m_1, m_2 and m_3 all even, is given by*

$$g(H_j) = 1 + \frac{M^{(j)}(j - 2 - m^{(j)})}{4}.$$

Then, we also established the genus of the j-cube, which is the repeated Cartesian product of K_2.

Theorem 5.1.7. *[1, 19, 28] The genus of the j-cube is given by*

$$g(Q_j) = 1 + 2^{j-3}(j - 4).$$

In Chapter 3, we have also presented some new results concerning the Cartesian product of the $2r$-cube $Q_i^{(2r)}$ with the repeated Cartesian product of even cycles G_j and with the repeated Cartesian product of even paths H_j. These results, found in Theorem 5.1.8 and Theorem 5.1.9, were submitted to a peer-reviewed journal for eventual publication.

Theorem 5.1.8. *The genus of $Q_i^{(2r)} \square G_j$ is given by*

$$g(Q_i^{(2r)} \square G_j) = 1 + M^{(j)} 2^{2i+j-2} r^i (j + ir - 2).$$

Theorem 5.1.9. *The genus of $Q_i^{(2r)} \square H_j$ is given by*

$$g(Q_i^{(2r)} \square H_j) = 1 + 2^{2i+j-2} r^i M^{(j)} (ir + j - m^{(j)} - 2).$$

The Cartesian product of the family of graphs $K_{1,3} \square P_{r+1}$ was also established, as stated in the theorem below. This result is also to be submitted to a peer-reviewed journal.

Theorem 5.1.10. *For $r \geq 1$, the genus of $K_{1,3} \square P_{r+1}$ is given by*

$$g(K_{1,3} \square P_{r+1}) = \left\lceil \frac{r-1}{2} \right\rceil.$$

Another result worth mentioning is the repeated direct product of j-cubes. Let $M^{(k)}$ denote $\prod_{i=1}^{k} j_i$, then the genus is given in the theorem below.

Theorem 5.1.11. *[31] For $i \in [k]$, let $G_i = Q_{j_i}$ with $j_i \geq 2$. Let $j = j_1 + \ldots + j_k$. Then, the genus of $\bigtimes_{i=1}^{k} Q_{j_i}$ is given by,*

$$g\left(\bigtimes_{i=1}^{k} Q_{j_i}\right) = 2^{j-3}(M^{(k)} - 4) + 2^{k-1}.$$

5.2 | Relationships

There are some asymptotic results proven by White [29] which relate the genus of a family of graphs to another. One such result concerns the genus of the Cartesian product of two k-partite graphs with the genus of a k-partite graph.

Theorem 5.2.1. *[29] For $s \geq r^{1+\epsilon}$, where $\epsilon > 0$ and s tending to infinity, $g(K_{k(r)} \square K_{k(s)}) \rightarrow krg(K_{k(s)})$, for all natural numbers $k > 1$.*

Proof. From Theorems 3.1.1 and 3.1.2

$$\begin{aligned} \max(krg(K_{k(s)}) + g(K_{k(r)}), ksg(K_{k(r)}) + g(K_{k(s)})) \\ \leq g(K_{k(r)} \square K_{k(s)}) \\ \leq krg(K_{k(s)}) + ksg(K_{k(r)}) + (k^2mn - k(m+n) + 1) \end{aligned} \tag{5.1}$$

Divide throughout by $krg(K_{k(s)})$ and take the limit as s approaches infinity.
Claim:

$$\lim_{s \to \infty} \frac{ng(K_{k(r)})}{mg(K_{k(s)})} = 0$$

.

Proof: By Lemma 1.2.9,

$$g(K_{k(s)}) \geq \frac{k(k-1)s^2 - 6ks + 12}{12}$$

and

$$\begin{aligned} g(K_{k(r)}) &\leq g(K_{kr}) \\ &= \lceil \frac{(kr-3)(kr-4)}{12} \rceil \\ &< \frac{k^2r^2 - 7kr + 24}{12} \end{aligned}$$

It then follows that

$$\begin{aligned} 0 &\leq \lim_{s \to \infty} \frac{sg(K_{k(r)})}{rg(K_{k(s)})} \\ &\leq \lim_{s \to \infty} \frac{s(k^2r^2 - 7kr + 24)}{r(k(k-1)s^2 - 6ks + 12)} = 0 \end{aligned}$$

■

The right-hand side of (5.1) can be evaluated to obtain:

$$\max\left(1, \frac{1}{kr}\right) \leq \lim_{s\to\infty} \frac{g(K_{k(r)\Box K_{k(s)}})}{krg(K_{k(s)})} \leq 1$$

Therefore,

$$\lim_{s\to\infty} \frac{g(K_{k(r)}\Box K_{k(s)})}{krg(K_{k(s)})} = 1.$$

□

As a consequence, for $k = 2$, we get a result for bipartite graphs.

Corollary 5.2.2. *For $s \geq r^{1+\epsilon}$, where $\epsilon > 0$ and s tending to infinity, $g(K_{r,r}\Box K_{s,s}) \to 2rg(K_{s,s})$.*

5.3 | Future Work

Whilst this study covered some of the most well-known and commonly used families of bipartite graphs, there are other graphs and graph products to be considered. Below we provide three possible extensions to this study to further investigate the problem of determining the genus of a graph.

5.3.1 | Other Families of Graphs

Apart from bipartite graphs, there are several other families of graphs with known genera. This includes the complete graph K_p, whose genus was found by Ringel [20] as a consequence of the "Map Colour Conjecture".

Theorem 5.3.1. *[20] The genus of the complete graph K_p, for $p \geq 3$, is given by*

$$g(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil.$$

White [28] also found an asymptotic relationship between the genus of a complete graph and the genus of the Cartesian product of two complete graphs, as stated in the theorem below.

Theorem 5.3.2. *[29] For $q \geq p^{1+\epsilon}$, where $\epsilon > 0$ and q tends to infinity, $g(K_p \square K_q) \rightarrow pg(K_q)$.*

5.3.2 | Other Operations on Graphs

The Cartesian and direct products are two of the main products on graphs, however, there are results on other graph products worth mentioning, namely the lexicographic product and the join of two graphs.

Definition 5.3.3. *[13] The* lexicographic product *of two graphs G and H is the graph $G \circ H$ whose vertex-set is $V(G) \times V(H)$ and a vertex (u, v) is adjacent to another vertex (u', v') if $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.*

Next, we present a general result on the lexicographic product of two graphs.

Theorem 5.3.4. *[5] If G is a nontrivial, connected graph on n vertices and m edges with minimum degree at least 2, and H is a graph on an even number of vertices $2r$, then*

$$g(G[H]) \leq r^2 m - rn + 1,$$

with equality if G is triangle-free.

As a consequence, this theorem can be applied to several families of graphs, including cycles, the j-cube and complete bipartite graphs. For another interesting result, we need the following definition.

Definition 5.3.5. *[4] The* join *of two vertex-disjoint graphs G and H, denoted by $G + H$, has vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.*

The next theorem states that there is a triangular embedding for two families of joins of graphs.

Theorem 5.3.6. *[16] There is an orientable embedding of $K_{36t+6} + \overline{K_{1+2i}}$ for $i \in \{1, \ldots, 12t\}$ and $t \geq 1$, and of $K_{36t+18} + \overline{K_{1+2i}}$ for $i \in \{1, \ldots, 12t + 4\}$ and $t \geq 0$.*

Another interesting result relates the genus of the join of two complete graphs to the genus of the complete bipartite graph.

Theorem 5.3.7. *[7] Suppose q is even and $q \geq 2$. Then $g(\overline{K_p} + K_q) = g(K_{p,q}) = \left\lceil \frac{(p-2)(q-2)}{4} \right\rceil$ for all $p \geq q$.*

5.3.3 | Open Problems

In Chapter 3, we established the genus of $K_{1,3} \square P_r$, but the genus of the Cartesian product $K_{1,s} \square P_r$ is still an open problem.

Problem 5.3.8. *What is the genus of $K_{1,s} \square P_r$, for any s and r?*

There is an upper bound of the genus for this family as the crossing number of this family of graphs is known [2], however, the genus may be significantly smaller.

Another interesting problem concerns the genus of the complete tripartite graph $K_{p,q,r}$ for $p \geq q \geq r \geq 1$ proposed by White [27], who provided the following conjecture.

Conjecture 5.3.9. *[27] For a complete tripartite graph $K_{p,q,r}$ for $p \geq q \geq r \geq 1$, the genus is given by*

$$g(K_{p,q,r}) = \left\lceil \frac{(p-2)(q+r-2)}{4} \right\rceil.$$

The survey by Wan [25] gives a list of these cases since a number of the cases of the conjecture above have been solved, however, it still remains as an open problem.

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Appendix: Paper

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The minimum orientable genus of the repeated Cartesian product of families graphs

Marietta Galea · John Baptist Gauci

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Abstract Determining the minimum genus of a graph is a fundamental optimisation problem in the study of network design and implementation as it gives a measure of non-planarity of graphs. In this paper, we are concerned with determining the smallest value of g such that a given graph G has an embedding on the orientable surface of genus g. In particular, we consider the Cartesian product of graphs since this is a well studied graph operation which is often used for modeling interconnection networks. The s-cube $Q_i^{(s)}$ is obtained by taking the repeated Cartesian product of i complete bipartite graphs $K_{s,s}$. We determine the genus of the Cartesian product of the $2r$-cube with the repeated Cartesian product of cycles and of the Cartesian product of the $2r$-cube with the repeated Cartesian product of paths.

Keywords genus · Cartesian product · complete bipartite graph · cycle · path

Mathematics Subject Classification (2020) 05C10 · 05C50

1 Introduction

Ever since their introduction by Sabidussi in 1959 [7], Cartesian product graphs have been applied in many areas, including in the study of interconnection networks and coding theory. Many important classes of graphs such as hypercubes, Hamming graphs, and prisms, are Cartesian products. This graph product provides a quick, effective and efficient way of constructing bigger graphs from smaller ones in such a way that the structure of the smaller graphs is preserved. Hence, Cartesian product graphs play a key role in the design and analysis of interconnection networks.

M. Galea
Department of Mathematics, University of Malta, Msida, Malta
E-mail: marietta.galea@gmail.com

J.B. Gauci
Department of Mathematics, University of Malta, Msida, Malta
E-mail: john-baptist.gauci@um.edu.mt

Various disciplines which deal with network design are interested in determining the optimal surface on which a network can be drawn without any of the links between its nodes intersecting each other. Mathematically, this problem revolves around determining the surface of minimum genus that permits a given graph to be drawn on it without edge-crossings. In computer science, the minimum genus of a graph is also a useful parameter in algorithm design since faster algorithms can be designed when the topological structure of a graph is known. These algorithms usually assume that the input graph is embedded in some surface, as for example in [1]. The problem of determining the minimum genus of a graph, albeit very relevant and applicable, has been proved to be substantially difficult from the theoretical, practical and structural perspectives. Garey and Johnson listed its complexity as one of the most important open problems in [2].

The problem of embedding graphs on surfaces other than the sphere dates back to the Map Colour Conjecture due to Heawood in 1890 [3]. Throughout this paper, a *surface* is a 2-manifold (that is, each point has a neighbourhood homeomorphic to a sphere) and compact (that is, closed and bounded). Surfaces can be classified into two categories: *orientable* and *nonorientable*. The former are obtained by adding *handles* to the sphere, whereas the latter are obtained from the sphere by adding *crosscaps*. In this work we restrict our attention to orientable surfaces S_g, where g is the *genus* of the surface, that is, the number of handles added to the sphere S_0. A handle is added to the sphere by removing two disjoint open discs and identifying their boundaries with the ends of a truncated cylinder.

An *embedding* of a graph G is a drawing of a graph on a surface such that no two edges cross. If every face in a drawing of G is homeomorphic to an open disc, the embedding is referred to as a *2-cell embedding*. Such an embedding in which each face has four sides is called a *quadrilateral embedding*. Given a graph G, if G has a 2-cell embedding on the surface S_g but not on the surface S_{g-1}, then G is said to be of *genus* g and we write $g(G) = g$.

Thomassen [8] showed that determining the genus of a graph is NP-complete. Nevertheless, results on the genus can be obtained by considering families of graphs with specific properties. One of the first results in this area is due to Ringel [6], who determined the genus of the complete bipartite graph $K_{m,n}$. White [9] considered Cartesian product graphs, where the *Cartesian product* $G_1 \Box G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1 \Box G_2) = \{(v_1, v_2) : v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\}$ and edge set $E(G_1 \Box G_2) = \{(v_1, v_2)(w_1, w_2) : (v_1 = w_1 \text{ and } v_2 w_2 \in E(G_2)) \text{ or } (v_1 w_1 \in E(G_1) \text{ and } v_2 = w_2)\}$. He determined the genus of the Cartesian product of the complete bipartite graph $K_{2m,2m}$ with $K_{2n,2n}$, for all natural numbers m and n. White also considered three families of graphs obtained by taking repeated Cartesian products, namely the repeated Cartesian product of paths, the repeated Cartesian product of cycles, and the repeated Cartesian product of complete bipartite graphs, respectively defined more precisely as follows:

$$H_j = H_{j-1} \Box P_{m_j} \text{where } j \geq 2 \text{ and } H_1 = P_{m_1},$$
$$G_j = G_{j-1} \Box C_{2m_j} \text{where } j \geq 2 \text{ and } G_1 = C_{2m_1},$$
$$Q_i^{(s)} = Q_{i-1}^{(s)} \Box K_{s,s} \text{where } i \geq 2 \text{ and } Q_1^{(s)} = K_{s,s}.$$

We remark that

(i) P_n and C_n denote the path on n vertices and the cycle on n vertices, respectively;

(ii) the well-known hypercube Q_n is H_n with $m_j = 2$ for all $j \in \{1, 2, \ldots, n\}$;

(iii) $Q_i^{(s)}$, also referred to at the s-cube, is a generalisation of the hypercube Q_n since $Q_n^{(1)} = Q_n$.

Pisanski [4] generalised White's result by considering the repeated Cartesian product of connected regular bipartite graphs on an even number of vertices. In particular, letting $G(n, d)$, for $n \geq d \geq 1$, denote a connected regular bipartite graph of degree d on $2n$ vertices, Pisanski determined the genus of the repeated Cartesian product of $G(n, d)\Box G_1(n_1, d_1)\Box G_2(n_2, d_2)\Box \ldots \Box\, G_m(n_m, d_m)$ provided that $\max\{d_1, d_2, \ldots, d_m\} \leq d \leq d_1 + d_2 + \ldots + d_m$.

The aim of this paper is to generalise further the work done on the genus of the repeated Cartesian product of graphs. In our main results, respectively stated in Theorem 4 and Theorem 5 hereunder, we determine the minimum genus of the Cartesian products $Q_i^{(2r)}\Box G_j$ and $Q_i^{(2r)}\Box H_j$, where i and j are positive integers. We remark that most of these cases are excluded from the result by Pisanski due to the condition imposed therein on the degrees, as highlighted later on in this work.

2 Basic results

An important and well known result about the Cartesian product of two graphs is given in the following theorem.

Theorem 1 *Let G and H be two non-empty graphs. The Cartesian product $G\Box H$ is bipartite if and only if G and H are bipartite.*

It follows immediately that the repeated Cartesian product of bipartite graphs is also bipartite.

A direct consequence of Euler's generalized formula for 2-cell embeddings of a bipartite graph in any surface is stated below.

Theorem 2 *If the bipartite graph G with n vertices and m edges has a quadrilateral embedding in a surface, then $g(G) = 1 + \frac{m}{4} - \frac{n}{2}$.*

In [9], White proved the following results.

Proposition 1 *[9]*

1. $g(H_j) = 1 + \frac{1}{4}\left(\prod_{k=1}^{j} m_k\right)\left(j - 2 - \sum_{k=1}^{j} \frac{1}{m_k}\right)$ *for $j \geq 3$ and even m_1, m_2, m_3;*
2. $g(G_j) = 1 + 2^{(j-2)}(j-2)\prod_{k=1}^{j} m_k$ *for $j \geq 2$;*
3. $g\left(Q_i^{(r)}\right) = 1 + 2^{i-3}r^i(i-4)$ *for r even and $i \geq 1$ or for $r \in \{1, 3\}$ and $i \geq 2$.*

Ringel [6] proved the following result, which we will need in our proofs.

Theorem 3 *[6] The complete bipartite graph $K_{2r,2r}$ has an embedding on a surface of genus $(r-1)^2$.*

Another useful result by White in [9] establishes a partition of a set of faces found in a quadrilateral embedding for $K_{2r,2r}$, given below.

Lemma 1 *[9] For a quadrilateral embedding of $K_{2r,2r}$, the set of $2r^2$ quadrilateral faces may be partitioned into $2r$ subsets of r faces each so that each subset of r faces contains all $4r$ vertices of the graph.*

3 The Cartesian product of the $2r$-cube with cycles

In this section we use a technique developed by Pisanski [5] and White [9], referred to as the White-Pisanski method, that constructs a quadrilateral embedding for the Cartesian product of two regular bipartite graphs.

Note that the number of vertices of $Q_i^{(2r)}$ is $2^{2i}r^i$ and the number of edges is $i2^{2i}r^{i+1}$, which can be easily derived using induction or the Handshaking Lemma. We first construct a quadrilateral embedding of $Q_i^{(2r)}\square\, C_{2s}$ and then derive the genus of this graph.

Lemma 2 *The genus of $Q_i^{(2r)}\square\, C_{2s}$ is given by*

$$g(Q_i^{(2r)}\square\, C_{2s}) = 1 + 2^{2i-1}sr^i(ir-1).$$

Proof First, we minimally embed a copy of $Q_i^{(2r)}$ as follows. By Theorem 1, $K_{2r,2r}\square\, K_{2r,2r}$ is a bipartite graph. Using Theorem 3, we embed $4r$ disjoint copies of $K_{2r,2r}$ in $4r$ surfaces of genus $(r-1)^2$ such that each embedding is quadrilateral. Assign an orientation to $2r$ of these copies and give the reverse orientation to the other half. This partition corresponds to the vertex-set partition of $K_{2r,2r}$. All that is left is to add edges between the $4r$ disjoint copies to acquire $K_{2r,2r}\square\, K_{2r,2r}$. Note that $4r$ edges are required to be added between each pair of oppositely oriented copies, which can be done by adding r handles, each carrying four edges. Such an operation will be referred to as a link. For each surface, there are $2r$ links to be made and by Lemma 1, each link is designated a subset of faces from the partition. We just need to check that after partitioning the faces, the corresponding subsets can be matched correctly. For $1 \leq \overleftarrow{j} \leq 2r$, each copy $\overleftarrow{j}$ of $K_{2r,2r}$ is embedded with common orientation, whilst for $1 \leq \overrightarrow{j} \leq 2r$ each copy $\overrightarrow{j}$ of $K_{2r,2r}$ is embedded with the reverse orientation. For some copy $\overleftarrow{j}$, its face partition consists of $2r$ subsets of r faces each. Match subset $\overleftarrow{k}$ of the face partition to subset $\overrightarrow{k}$ in copy $\overrightarrow{j} + \overrightarrow{k}, 1 \leq \overrightarrow{j} + \overrightarrow{k} \leq 2r \pmod{2r}$. If $\overrightarrow{j} + \overrightarrow{k} = \overrightarrow{j}' + \overrightarrow{k}'$ with $\overrightarrow{k} = \overrightarrow{k}'$, then, $\overleftarrow{j} = \overleftarrow{j}'$, therefore, each subset of the partition has exactly one handle attached at each face in that subset. Each handle carries a maximum of four edges, so that each new face formed is quadrilateral. This process is repeated by successively introducing the complete bipartite graph $K_{2r,2r}$ until a quadrilateral embedding of $Q_i^{(2r)}$ is obtained.

Now, to embed $Q_i^{(2r)}\square\, C_{2s}$, embed $2s$ copies of $Q_i^{(2r)}$ as described above. We assign s many copies one orientation and the other s copies the reverse orientation, corresponding to the vertex-set partition of C_{2s}. From each copy of $Q_i^{(2r)}$, two links are to be made, both to copies of reverse orientation. For each link, $2^{2i}r^i$ edges are to be added and these are passed over $2^{2i-2}r^i$ handles with each handle carrying four edges. By the construction given above and Lemma 1, each copy of $Q_i^{(2r)}$ has $2r$ disjoint sets of $2^{2i-2}r^i$ mutually vertex-disjoint quadrilateral faces that cover all $2^{2i}r^i$ vertices of $Q_i^{(2r)}$. Since only two links are to be made from each copy of $Q_i^{(2r)}$, we choose any two of the available sets. The number of quadrilateral faces in each set is precisely the amount required to perform a link, as previously described. Since the embedding of each copy of $Q_i^{(2r)}$ is quadrilateral and each insertion of a handle introduces more quadrilateral faces, the obtained embedding of $Q_i^{(2r)}\square\, C_{2s}$ is quadrilateral. Applying Theorem 2, we get

$$\begin{aligned} g(Q_i^{(2r)}\square\, C_{2s}) &= 1 + \frac{2sir^{i+1}2^{2i} + 2sr^i 2^{2i}}{4} - \frac{2sr^i 2^{2i}}{2} \\ &= 1 + 2^{2i-1} sr^i (ir - 1). \end{aligned}$$

□

Let $M^{(j)} = \prod_{k=1}^{j} m_k$. Note that the number of vertices of G_j is $2^j M^{(j)}$ and the number of edges is $j2^j M^{(j)}$, which follows immediately by induction. We are now in a position to construct a quadrilateral embedding of $Q_i^{(2r)}\square\, G_j$ and, consequently, determine the genus of this graph.

Theorem 4 *The genus of $Q_i^{(2r)}\square\, G_j$ is given by*

$$g(Q_i^{(2r)}\square\, G_j) = 1 + M^{(j)} 2^{2i+j-2} r^i (j + ir - 2).$$

Proof We let G denote $Q_i^{(2r)}\square\, G_j$. The proof proceeds by induction on j, where the property $P(j)$ is as follows: There is a quadrilateral embedding of G, with the number of faces $f_j = r^i 2^{2i+j-1} M^{(j)}(j + ir)$, including two disjoint sets each of $r^i 2^{2i+j-2} M^{(j)}$ mutually vertex-disjoint quadrilateral faces each such that each set contains all the $r^i 2^{2i+j} M^{(j)}$ vertices of G.

For $j = 1$, the quadrilateral embedding of $Q_i^{(2r)}\square\, C_{2m_1}$ is given by Lemma 2. The number of faces f_1 is given by $f_1 = \frac{1}{2}\left|E(Q_i^{(2r)}\square\, C_{2m_1})\right|$, and thus

$$\begin{aligned} f_1 &= \frac{(2m_1)(2^{2i}r^i) + (2m_1)(i2^{2i}r^{i+1})}{2} \\ &= m_1 2^{2i} r^i + im_1 2^{2i} r^{i+1} \\ &= 2^{2i} m_1 r^i (1 + ir). \end{aligned}$$

which conforms to the number of expected faces as stated by $P(1)$. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links, and the second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, each disjoint set would contain $2m_1(2^{2i-2}r^i) = 2^{2i-1}m_1 r^i$ quadrilateral faces. This follows since half of the $2m_1$ links are chosen, where each link has $2^{2i-2}r^i$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of $Q_i^{(2r)}\square\, C_{2m_1}$. Therefore, $P(1)$ holds.

Assuming $P(j)$ is true, we will show that $P(j+1)$ also holds. The graph we are considering is $G' = Q_i^{(2r)}\square\, G_{j+1}$. First, minimally embed $2m_{j+1}$ copies of G as described by $P(j)$. Assign m_{j+1} many copies one orientation and the other m_{j+1} copies the reverse orientation. This partition corresponds to the vertex-set partition of $C_{2m_{j+1}}$. From each copy of G, two links are to be made, both to copies of reverse orientation. The number of vertices of G is $r^i 2^{2i+j} M^{(j)}$, and thus we need to pass that many edges over a link joining two copies of G with different orientations. Therefore, the number of handles required for each link is $\frac{1}{4}\left(r^i 2^{2i+j} M^{(j)}\right)$ since each handle carries four edges. Using $P(j)$, we have two disjoint sets each of $r^i 2^{2i+j-2} M^{(j)}$ mutually vertex-disjoint quadrilateral faces and thus we have a sufficient number of faces such that we can construct these links using $r^i 2^{2i+j-2} M^{(j)}$ handles, where each handle carries four edges so that each new face formed is a quadrilateral. In this manner, the graph G' is embedded after constructing the $2m_{j+1}$ links.

The number of faces f_{j+1} in G' is $f_{j+1} = 2m_{j+1} f_j + \tilde{f}$, where $\tilde{f}$ denotes the number of new faces introduced by the handles. Note that $\tilde{f} = 4m_{j+1}(r^i 2^{2i+j-2} M^{(j)})$ since we have $2m_{j+1}$ links, with each link requiring $r^i 2^{2i+j-2} M^{(j)}$ handles and the difference in the number of faces is two per handle. Therefore,

$$\begin{aligned} f_{j+1} &= 2m_{j+1}(r^i 2^{2i+j-1} M^{(j)}(j+ir)) + 4m_{j+1}(r^i 2^{2i+j-2} M^{(j)}) \\ &= r^i 2^{2i+j} M^{(j+1)}(j+ir) + r^i 2^{2i+j} M^{(j+1)} \\ &= r^i 2^{2i+j} M^{(j+1)}(j+ir+1). \end{aligned}$$

This satisfies the expected number of faces as stated by $P(j+1)$. Next, we have to construct the two disjoint sets of faces. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links, and the second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, each disjoint set would contain $2m_{j+1}(r^i 2^{2i+j-2} M^{(j)}) = 2^{2i+j-1} r^i M^{(j+1)}$ quadrilateral faces. This follows since half of the $2s$ links are chosen, where each link has $r^i 2^{2i+j-2} M^{(j)}$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of G'. Therefore, $P(j+1)$ follows from $P(j)$ and since G is bipartite by Theorem 1, we calculate the genus using Theorem 2 to get

$$\begin{aligned} g(Q_i^{(2r)} \square\, G_j) &= 1 + \frac{r^i 2^{2i+j} M^{(j)}(j+ir)}{4} - \frac{2^{2i+j} r^i M^{(j)}}{2} \\ &= 1 + M^{(j)} 2^{2i+j-2} r^i (j+ir-2). \end{aligned}$$

□

As a direct corollary of the above result, we establish the genus of $K_{2r,2r} \square G_j$, addressing the case when $r > j$ which is excluded from Pisanski's result.

Corollary 1 *The genus of $K_{2r,2r} \square G_j$ is given by*

$$g(K_{2r,2r} \square\, G_j) = 1 + r 2^j M^{(j)}(j+r-2).$$

4 The Cartesian product of the 2*r*-cube with paths

Using Lemma 2, we can establish the genus of $Q_i^{(2r)} \square\, P_{2s}$ as follows.

Lemma 3 *The genus of $Q_i^{(2r)} \square\, P_{2s}$ is given by*

$$g(Q_i^{(2r)} \square\, P_{2s}) = 1 + 2^{2i-2} r^i (2s(ir-1)-1).$$

Proof We let G denote $Q_i^{(2r)}\square\, P_{2s}$ and note that G is bipartite by Theorem 1. We start by first embedding $G' = Q_i^{(2r)}\square\, C_{2s}$ as given in Lemma 2. To obtain G, we only need to remove $4^i r^i$ edges from G' between a copy of $Q_i^{(2r)}$ with one orientation and another copy of reverse orientation. The only way these two copies are connected is through $2^{2i-2}r^i$ handles, therefore, we remove these $2^{2i-2}r^i$ handles and the edges on them. The faces we have reintroduced are quadrilateral, since the embedding of G' is quadrilateral, producing a quadrilateral embedding for G. The genus of G is given by,

$$\begin{aligned} g(Q_i^{(2r)}\square\, P_{2s}) &= 1 + 2^{2i-1}sr^i(ir-1) - 2^{2i-2}r^i \\ &= 1 + 2^{2i-2}r^i(2s(ir-1)-1). \end{aligned}$$

□

Similarly to the repeated product of even cycles, we denote by $M^{(j)}$ the product $\prod_{k=1}^{j} m_k$ and $m^{(j)}$ denotes $\sum_{k=1}^{j}\frac{1}{2m_k}$. Using Lemma 3, we establish the genus of $Q_i^{(2r)}\square\, H_j$.

Theorem 5 *The genus of $Q_i^{(2r)}\square\, H_j$ is given by*

$$g(Q_i^{(2r)}\square\, H_j) = 1 + 2^{2i+j-2}r^i M^{(j)}(ir + j - m^{(j)} - 2).$$

Proof We let G denote $Q_i^{(2r)}\square\, H_j$. The proof proceeds by induction on j, where the property $P(j)$ is as follows: There is a quadrilateral embedding of G, with $f_j = 2^{2i+j-1}r^i M^{(j)}(j + ir - m^{(j)})$, including two disjoint sets of $r^i 2^{2i+j-2}M^{(j)}$ mutually vertex-disjoint quadrilateral faces each, such that each set contains all the $r^i 2^{2i+j}M^{(j)}$ vertices of G.

For $j = 1$, the quadrilateral embedding of $Q_i^{(2r)}\square\, P_{2m_1}$ is given by Lemma 3. The number of faces f_1 is given by $f_1 = \frac{1}{2}\left|E(Q_i^{(2r)}\square\, P_{2m_1})\right|$, and thus

$$\begin{aligned} f_1 &= \frac{(2m_1-1)(2^{2i}r^i) + (2m_1)(i2^{2i}r^{i+1})}{2} \\ &= 2^{2i-1}r^i(2m_1 - 1 + 2m_1 ir) \\ &= 2^{2i}m_1 r^i\left(1 + ir - \tfrac{1}{2m_1}\right). \end{aligned}$$

which conforms to the number of expected faces as stated by $P(1)$. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links, and the second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, each disjoint set would contain $2m_1(2^{2i-2}r^i) = 2^{2i-1}r^i m_1$ quadrilateral faces. This follows since from $2m_1 - 1$ links, $\lceil\frac{2m_1-1}{2}\rceil$ are chosen, where each link has $2^{2i-2}r^i$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of $Q_i^{(2r)}\square\, P_{2m_1}$, therefore, $P(1)$ holds.

Assuming $P(j)$ is true, we will show that $P(j+1)$ also holds. The graph we are considering is $G' = Q_i^{(2r)}\square\, H_{j+1}$. First, minimally embed $2m_{j+1}$ copies of G' as described by $P(j)$. Assign m_{j+1} many copies one orientation and the other m_{j+1} copies the reverse orientation. This partition corresponds to the vertex-set partition of $P_{2m_{j+1}}$. From each copy of common orientation, except the end copy, of G, two links are to be made, all to copies of reverse orientation. We can construct these links using $r^i 2^{2i+j-2} M^{(j)}$ handles, where each handle carries four edges so that each new face formed is a quadrilateral. In this manner, the graph G' is embedded after constructing the $2m_{j+1}-1$ links.

The number of faces f_{j+1} in G' is $f_{j+1} = 2m_{j+1} f_j + \tilde{f}$, where $\tilde{f}$ denotes the number of new faces introduced by the handles. Note that $\tilde{f} = 2(2m_{j+1}-1)(r^i 2^{2i+j-2} M^{(j)})$ since we have $2m_{j+1}-1$ links, with each link requiring $r^i 2^{2i+j-2} M^{(j)}$ handles and the difference in the number of faces is two per handle. Therefore,

$$\begin{aligned}
f_{j+1} &= 2m_{j+1}(2^{2i+j-1} r^i M^{(j)}(j + ir - m^{(j)})) + (2m_{j+1}-1)(r^i 2^{2i+j-1} M^{(j)}) \\
&= r^i 2^{2i+j} M^{(j+1)} \left(ir + j - m^{(j)} + \frac{2m_{j+1}-1}{2m_{j+1}} \right) \\
&= r^i 2^{2i+j} M^{(j+1)} (ir + j + 1 - m^{(j+1)}).
\end{aligned}$$

This satisfies the expected number of faces as stated by $P(j+1)$. Next, we have to construct the two disjoint sets of faces. The first disjoint set of faces may be chosen by selecting opposite faces from alternate links. The second set of faces may be formed by choosing the remaining faces from the same handles in each chosen link. Therefore, the disjoint set would contain $2\left\lceil \frac{2m_{j+1}-1}{2} \right\rceil (r^i 2^{2i+j-2} M^{(j)}) = m_{j+1} 2^{2i+j-1} r^i M^{(j)} = 2^{2i+j-1} r^i M^{(j+1)}$ quadrilateral faces. This follows since half of the $2m_{j+1}-1$ links are chosen, where each link has $r^i 2^{2i+j-2} M^{(j)}$ handles and only two faces are chosen from each handle. The faces in each set are mutually vertex-disjoint and each set contains all the vertices of G'. Therefore, $P(j+1)$ follows from $P(j)$ and since G is bipartite by Theorem 1, we calculate the genus using Theorem 2 to get

$$\begin{aligned}
g(Q_i^{(2r)} \square H_j) &= 1 + \frac{r^i 2^{2i+j} M^{(j)} (j + ir - m^{(j)})}{4} - \frac{2^{2i+j} r^i M^{(j)}}{2} \\
&= 1 + r^i 2^{2i+j-2} M^{(j)} (j + ir - m^{(j)}) - 2^{2i+j-1} r^i M^{(j)}. \\
&= 1 + 2^{2i+j-2} r^i M^{(j)} (ir + j - m^{(j)} - 2).
\end{aligned}$$

□

We remark that Pisanski's result also excludes Theorem 5 since the path is not a regular graph.

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